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MEAN FIELD ELECTRODYNAMICS AND DYNAMO THEORIES OF PLANETARY MAGNETIC FIELDS

by

© JOHN MICHAEL GILLILAND

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled MEAN FIELD ELECTRODYNAMICS AND DYNAMO THEORIES OF PLANETARY MAGNETIC FIELDS submitted by JOHN MICHAEL GILLILAND in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Geophysics.

Dedicated with gratitude and affection

to my parents

DEAN AND MRS. HENRY C. GILLILAND

and to my wife

MARGARET,

without whose continuing encouragement and loyal support

this thesis would not have been written.

ABSTRACT

In Chapter 1 homogeneous dynamo theory is reviewed Present observational knowledge of astrophysical magnetic fields is summarized, and is shown to provide less support for Schuster's hypothesis than the data presented by *Warwick (1971)*.

In Chapter 2 a review of "mean field electrodynamics" is presented, and the dispersion relation for "wave" mean fields is discussed. A new terminology is suggested for several types of stationary, homogeneous turbulence.

In Chapter 3 it is shown that, to a first approximation, stationary, homogeneous turbulence whose average properties are invariant under space-time inversion (PT-invariant turbulence) cannot support dynamo action in an incompressible fluid. This result is in direct contradiction to the work of *Lerche and Low (1971)*. However, "mirror-symmetric" turbulence which is not PT-invariant cannot definitely be ruled out as a source of dynamo action.

The decay of "wave" mean fields in the presence of PT-invariant turbulence is studied. It is shown that spatially periodic mean fields can exist only if the correlation tensor of the turbulence satisfies certain conditions. Conditions are also established for validity of the *Rädler (1968)* expression for "turbulent magnetic diffusivity", when the mean field does not oscillate with time. When the mean

field is oscillatory, Rädler's techniques are not useful. Numerical results are presented for oscillating mean fields, showing the relationship between mean field frequency and wavelength and the properties of the turbulence. The possibility of a "sporadic helicity" dynamo is discussed.

In Chapter 4 a technique is presented for dealing with nonstationary, inhomogeneous turbulence within the framework of mean field electrodynamics. The technique is applied to the kinematic dynamo problem.

In Chapter 5 temporal variations of magnetic fields are discussed. The "$\alpha^2(r)$" kinematic dynamo in a spherical shell is studied in detail. It is shown that the dependence of α on r near the spherical boundary can control the time behaviour of the external magnetic field. Integral properties of the dynamo equations for more general velocity distributions are discussed, and the possibility of "boundary-layer control" is considered in detail. It is found that in the geodynamo, temporal variations of the dipole moment on scales less than 10^4 years may be explained by boundary-layer phenomena.

In Chapter 6 the hydromagnetic dynamo problem is studied and the likelihood of "boundary-layer control" in the geodynamo is assessed. It is shown that "dipole wobble" can be explained as an effect of the slow, systematic decrease of the Earth's rate of rotation, if the kinematic viscosity at the core-mantle interface is approximately

$1\text{-}2\ m^2/sec$. Non-periodic variations of the axial dipole moment on the time scale of geomagnetic polarity transitions can also be explained by this model.

The effects of inhomogeneous, locally isotropic, turbulent forces in the Earth's core are considered. The characteristic turbulent length scale and the ratio of the diffusion time on this length scale to the effective turbulent time scale are important parameters. In most cases, a rotation-dependent α-effect is dominant when the mean magnetic field is small.

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1.1 Astrophysical magnetic fields

1.1.1 Introduction, and summary of observations

Although large-scale magnetic fields have long been an important feature of both planetary physics and astrophysics, it is only in recent years that significant progress has been made in understanding how these fields are generated and maintained. In part, this improved understanding is due to a marked increase in the amount of observational data available on astrophysical magnetic fields. The fields of interest fall into four broad categories.

a. "Weak" astrophysical magnetic fields (Table 1)

This category includes intergalactic, interstellar, and interplanetary magnetic fields. Of these, perhaps the most interesting from a theoretical point of view is the field of order $3\text{-}4 \times 10^{-6}$ G in the rotating, gaseous disk of the Galaxy.*

b. Planetary magnetic fields (Tables 2 and 3)

This category includes the poloidal surface fields of the Earth (~0.6 G at the magnetic poles) and Jupiter

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* In this thesis, SI units will be used as a general rule. However, astrophysical magnetic fields will often be quoted in gauss ($1 \text{ G} = 10^{-4}$ tesla). See Appendix 1 for a summary of the SI system of units.

(10-50 G at the magnetic poles). None of the other planets have been observed to have magnetic fields, but Saturn, Uranus, and Neptune could have undetected fields of the order of a few gauss (see section 1.1.3). Mars, Venus, and Mercury are thought to be "non-magnetic" planets.

c. Solar magnetic fields (Table 4)

This category includes the general background solar field (of the order of a few gauss), and the much stronger local fields (up to several thousand gauss) observed in magnetically active regions.

d. Stellar magnetic fields (Table 5)

This category includes the fields observed in some bright stars (of order 10^2 G); the stronger fields (of order 10^3-10^4 G) inferred from spectroscopic observation of the so-called "magnetic stars"; the still stronger fields (of order 10^7 G) observed in magnetic white dwarfs; and the extremely strong fields (of order 10^{12} G or higher) inferred from observations of pulsars.

In this thesis we shall be concerned mainly with fields of planetary type, although the techniques employed can be applied to other types of field as well.

It should be noted that the numbers in parentheses in Tables 1-8 refer to a special section of footnotes, to be

found at the end of section 1.1.3, on pages 21-23. These footnotes give references to the literature on astrophysical magnetic fields.

TABLE 1 - WEAK ASTROPHYSICAL MAGNETIC FIELDS

Location	Strength ($\mu G = 10^{-10} T$)	Spatial structure	Possible source(s)
Intergalactic space (1)	<0.02	Disordered	a) "Primeval" fields b) Stretched-out galactic fields
Galactic corona (2)	0.3-1	Unknown	a) Stretched-out galactic fields b) Compressed intergalactic fields
Galactic disk (3)	3-5	Probably a spiral structure with lines of force directed along the galactic arms. There is a fluctuating component, probably of the same magnitude as the spiral field.	a) Dynamo action due to nonuniform rotation and cyclonic turbulence b) Stretched-out stellar fields c) Compressed intergalactic fields
Supernova remnants (4)	40-300	Unknown	a) Stretched-out pulsar fields b) Compressed interstellar fields
Other galaxies (5)	up to 1000	Unknown	Dynamo action(??)
Quiet interplanetary space (6)	5000 (at 1 AU)	Spiral structure centered on the Sun, extending out to 30 AU or more. There is a superimposed fluctuating component.	Stretched-out solar fields

TABLE 2 - GEOMAGNETIC FIELDS OF INTERNAL ORIGIN

Location	Strength ($G=10^{-4}$ T)	Spatial structure	Probable source
Surface (7)		Mainly dipolar, with dipole axis inclined about 11.5° to axis of rotation at the present time. Field averages to an axial dipole within 2.7×10^4 yr., and to a geocentric axial dipole within 2×10^6 yr. Average field has been an axial dipole for at least $2\text{-}3 \times 10^9$ yr.	Dynamo action in the fluid core of the Earth. (But see Lyttleton, 1970!)
dipole field (at magnetic poles	0.6		
non-dipole field	0.02		
Core-mantle interface			Dynamo action in the fluid core
poloidal [extrapolation of surface fields (8)]	5-6(?)	Probably mainly poloidal, with the non-dipole field of the same order as or greater than the dipole field.	
toroidal [estimate (9)]	0.2	Small toroidal component due to non-zero conductivity of lower mantle	
Core [various estimates (10)]	50-500	Probably mainly toroidal	Dynamo action in the fluid core

TABLE 3 - PLANETARY AND LUNAR MAGNETIC FIELDS

Location	Strength ($G=10^{-4}$ T)	Spatial structure	Possible source(s)
Jupiter (11) surface (at magnetic poles)	10-50	Surface field is a centred dipole inclined about 11° to axis of rotation (best fit to deca- and decimetric radio observations). Non-dipole field is much smaller in proportion than that for the Earth. Interior field is probably mainly toroidal.	a) Dynamo action in fluid core (assuming one exists)
interior (estimate)	>1000		b) Dynamo action in lower atmosphere (but conductivity is probably too low)
Saturn - surface (12)	$\lesssim 1$(?)	If a field exists, it is probably similar to that of Jupiter.	Dynamo action in lower atmosphere(?) - fluid core probably not present(?)
Uranus - surface Neptune - surface (13)	$\lesssim 5$(??)	If fields exist, they are probably similar to that of Jupiter.	Dynamo action in fluid core, if one exists, or in atmosphere (?)
Mars - surface (14)	<0.001	(Mariner 4 observations)	Any field present is likely to be a compressed interplanetary field.
Venus - surface (15)	<0.0004	(Venera IV observations)	
Mercury - surface (16)	little or none	(Estimate)	
Lunar surface			a) Ancient dynamo action in fluid core(?) b) Long-term immersion in enhanced geomagnetic field at some time in past(?) c) High-velocity meteorid impacts(?)
magnetizing field implied by remanent magnetization (17)	0.02-0.05	Unknown (may be complicated) [Possible sources listed at right apply to this field]	
global dipole field (at magnetic poles) (18)	$<2 \times 10^{-6}$	Dipolar	

TABLE 4 - SOLAR MAGNETIC FIELDS

Location	Strength ($G = 10^{-4}T$)	Spatial structure	Possible source(s)
Global fields in photosphere			
axial field (19)	1-2	Variable. Usually taken to be roughly dipolar within 35° of the poles.	a) Dynamo action in the solar convection zone
"sector structure" (20)	$\lesssim 3$	Variable. Indirect measurements imply a persistent quadrupole moment, partly obscured during active portions of the solar cycle. There is also some evidence that the equatorial dipole at times dominates the axial dipole.	b) "Deep-seated magnetic sources"
Local fields in photosphere (21)	1-5000	Complex, variable, filamentary structure.	Eruption and compression of strong internal fields
Internal fields [theoretical estimates (22)]	several hundred to a thousand	In convection-zone dynamo theories, internal fields are mainly toroidal, and are confined to the region within $0.1R_{\odot}$ from the solar surface. "Deep-seated field" theory requires a much deeper penetration of strong fields.	Dynamo action in solar convection zone "Deep-seated magnetic sources"

TABLE 5 - STELLAR MAGNETIC FIELDS

Location	Strength (G= 10^{-4}T)	Spatial structure	Possible source(s)
Some bright stars surface fields (23)	30-300	Dipolar(?)	Dynamo action in stellar interior(?)
Sirius-like stars surface fields (24)	500-1000	Dipolar(?)	Dynamo action in stellar interior(?)
Ap stars (25) surface fields	1000-34000	Dipolar, with dipole axis inclined to axis of rotation (typically ~80°; some ~0°), OR non-axisymmetric poloidal field symmetric about the equatorial plane.	a) Dynamo action in stellar interior b) Compressed "fossil" fields
interior fields (estimated)	up to 10^{10} (?)	Probably mainly toroidal	
White dwarfs (26)	<1-2 x 10^{5}	If fields exist, they may be dipolar.	a) Compressed "fossil" fields b) Dynamo action(??)
Magnetic white dwarfs (27) surface fields	10^{6}-10^{8}	Axisymmetric or inclined dipole(?)	
interior fields (estimated)	up to 10^{12} (?)	Probably mainly toroidal	
Pulsars (28) surface fields	10^{11}-10^{14}	Dipolar, with dipole axis inclined to axis of rotation	Compressed "fossil" fields (possibly from magnetic white dwarf stage). Dynamo action is not possible in present theoretical models since convective fluid motions damp out within seconds of neutron star formation.
interior fields (estimated)	>10^{16}(?)	Probably mainly toroidal	

1.1.2 Dynamo theory and astrophysical magnetic fields

The connecting link between the various types of magnetic field mentioned in section 1.1.1 is the presence of a conducting fluid medium - e.g. planetary fluid cores, conducting planetary atmospheres, stellar atmospheres, and the interstellar gas. This observation leads to the conjecture, first advanced by *Larmor (1919)*, that the fields are hydromagnetic in origin. Evidence in support of this conjecture is provided by the fact that some of the fields vary in a complicated way with time. The Earth's magnetic field, for example, has existed in roughly its present form for at least 2.5-2.7 x 10^9 years *(McElhinny and Evans, 1968)*, but archaeo- and palaeomagnetic observations indicate that the field fluctuates on a wide range of time scales, and reverses its polarity at irregular intervals *(Bullard, 1968; Braginskii, 1970b, 1971; McElhinny, 1971)*. These considerations effectively rule out any possibility that the geomagnetic field is due to permanent magnetism *(Bullard, 1948, 1949; Rikitake, 1966a, p. 13ff.)*, and make it highly unlikely that the field is due to intrinsic "rotational" magnetism (see section 1.1.3). Attention is therefore focussed on electric currents in the Earth's fluid core as the source of the field. The most likely source of the electromotive force needed to maintain these currents for times long compared with the ohmic decay time (10^4-10^5 years) is the motion of core material across the geomagnetic lines of

force. The study of this process, in which the currents generated reinforce the magnetic field which gives rise to the driving e.m.f., is known as the "homogeneous dynamo problem".

Not all astrophysical magnetic fields are maintained by "dynamo action". In pulsars (neutron stars), for example, "...most internal motions except for axially symmetric differential rotation are damped out...within a few seconds after [the] neutron star is formed" *(Ruderman, 1972)*. Differential rotation of this type will convert poloidal lines of force into toroidal lines of force, and build up extremely large internal toroidal fields, but the motion cannot regenerate the poloidal field from the toroidal *(Elsasser, 1947, 1950)*. Dynamo action will therefore not occur.

It is generally thought that "...the initial magnetic field of a pulsar is...a compressed fossil field conserving the flux already present in its parent star core before collapse" *(Ruderman, 1972)*. In support of this hypothesis it is argued that the product BR^2 (where B is a typical poloidal magnetic flux density, and R the stellar radius) is approximately the same for magnetic stars, magnetic white dwarfs, and pulsars. The evidence in support of this claim is summarized in the first column of Table 7. Values of BR^2 for Ap stars, magnetic white dwarfs, and pulsars range over three orders of magnitude; however, the ranges for

magnetic white dwarfs and pulsars are very nearly the same. One difficulty with the argument is the scarcity of magnetic white dwarfs. Only four of the more than fifty white dwarfs which have been identified have observable magnetic fields. Any fields present in the "normal" white dwarfs must be less than $1\text{-}2 \times 10^5$ G *(Angel and Landstreet, 1970; Preston, 1970; Trimble and Greenstein, 1972)*. As indicated in Table 7, fields of this size are at least one order of magnitude too small to become pulsar fields under "flux-conserving compression".

It is argued by some workers *(e.g. Mestel, 1971)* that all large stellar fields are compressed "fossil fields". However, dynamo models have been proposed for magnetic stars *(e.g. Krause, 1971)*. "Small" stellar fields like that of the Sun, on the other hand, are generally thought to be due to dynamo action because of their extreme complexity and variability. However, some authors *(e.g. Piddington, 1972c)* feel that dynamo action is an insufficient explanation even for solar magnetic fields.

A similar situation exists with regard to the Galactic magnetic field. One school of thought *(e.g. Parker, 1969a, 1971a)* claims that the field in the galactic disk is maintained by dynamo action associated with turbulent motion, while another *(e.g. Piddington, 1972a,b)* argues that the field is more likely to be a compressed "primeval" field.

The remaining class of fields - planetary magnetic fields - is almost universally thought to be generated by dynamo action *(see, however, Lyttleton, 1970, and section 1.1.3)*. In the remainder of this chapter we shall consider the homogeneous dynamo theory as it applies to planetary fields, with particular emphasis on the magnetic field of the Earth.

TABLE 6 - QUANTITIES OF INTEREST FOR MAGNETIC STARS

Star	Rotation period	Surface field ($G = 10^{-4}T$)	$R/R_\odot$	$M/M_\odot$	I ($kg\text{-}m^2$)
Sun (29)	26d	1	1	1	6×10^{46}
γCyg (30)	2-3d(??)	200	70	10	
Sirius (30)	9d(??)	500	2.2	3	
Ap stars (31)	[1.7-2500]d	$[1-34] \times 10^3$	[1-4]	[1-5]	
typical values	6.5d	2.5×10^3	3.2	3	
White dwarfs (32)		$<[1-2] \times 10^5$			
Magnetic white dwarfs (32)	?	$[10^6-10^8]$	[0.0089-0.013]	[0.65-0.87]	
typical values		10^7	0.011	0.76	
Rotating magnetic white dwarfs [one known] (33)	1.34d	10^7	0.013(?)	[1-4](?)	
Pulsars (34)	[0.03-3.7]s	$[0.5-50] \times 10^{12}$	$[2-1] \times 10^{-5}$	[o.15-2]	$[0.7-7] \times 10^{37}$
Crab pulsar	0.033s	2×10^{12}	1.7×10^{-5}	0.3	0.9×10^{37}

M = stellar mass
R = stellar radius
I = moment of inertia

Subscript "⊙" refers to values for Sun.

TABLE 7 - FIELD-COMPRESSION HYPOTHESIS, ANGULAR MOMENTUM, AND DIPOLE MOMENT IN STELLAR BODIES

Star	$[B/B_\odot][R/R_\odot]^2$	$[J/T^{(1)}][J_\odot/T_\odot^{(1)}]^{-1}$	$[J/T^{(1)}][J_E/T_E^{(1)}]^{-1}$
Sun	1	1	13.2
γCyg	980×10^3	$\sim 8 \times 10^{-3}$(?)	0.11(?)
Sirius	2.4×10^3	6×10^{-3}(?)	0.08(?)
Ap stars	$[1-50] \times 10^3$	$[1 \times 10^{-7} - 8 \times 10^{-2}]$	$[1.5 \times 10^{-6} - 1]$
typical values	26×10^3	2×10^{-3}	0.02
White dwarfs	$<0.02 \times 10^3$	?	?
Magnetic white dwarfs	$[0.08-18] \times 10^3$		
typical values	1×10^3	$\sim 5 \times 10^{-4}$(?)	~0.007(?)
Pulsars (35)	<u>all</u> $[0.05-5] \times 10^3$ <u>most</u> $[0.17-1.5] \times 10^3$	$[1.5 \times 10^{-4} - 1.5 \times 10^2]$	$[2 \times 10^{-3} - 2 \times 10^3]$
Crab pulsar	0.6×10^3	1×10^{-2}	0.1

J = angular momentum; M = mass; R = radius; B = surface flux density

$T^{(1)}$ = magnetic dipole moment. Subscript "⊙" refers to Sun; subscript "E" to Earth.

$B_\odot R_\odot^2 \simeq 4.9 \times 10^{13}\ T \cdot m^2$

$J_\odot \simeq 1.7 \times 10^{41}\ m^2 \cdot kg \cdot s^{-1}$

$T_\odot^{(1)} \simeq 1.7 \times 10^{29}\ A \cdot m^2$

$$JT_\odot^{(1)}/J_\odot T^{(1)} \sim (I/I_\odot)(\Omega/\Omega_\odot)/(B/B_\odot)(R/R_\odot)^3$$
$$\sim (M/M_\odot)(\Omega/\Omega_\odot)/(B/B_\odot)(R/R_\odot)$$

where Ω = rotation frequency, I = moment of inertia

TABLE 8 - ANGULAR MOMENTA, DIPOLE MOMENTS, AND SURFACE FIELDS IN PLANETARY BODIES

Body	$[R/R_E]^3$	J/J_E	$[T^{(1)}/T_E^{(1)}]_{obs}$	$JT_E^{(1)}/J_ET^{(1)}$	$[B/B_E]_{obs}$	$[B/B_E]_{pred}$
Jupiter(36)	1.4×10^3	7.3×10^4	$[0.5-1] \times 10^5$	[0.7-1.5]	[36-72]	52
Saturn(37)	8.5×10^2	1.2×10^4	-	-	-	14
Uranus(38)	5.2×10^1	3.3×10^2	-	-	-	9
Neptune(38)	4.3×10^1	4.0×10^2	-	-	-	6
Earth	1	1	1	1	1	1
Venus(39)	8.8×10^{-1}	$\sim 3 \times 10^{-3}$	$<1 \times 10^{-3}$	>3	$<1 \times 10^{-3}$	0.03
Mars(40)	1.5×10^{-1}	3.5×10^{-2}	$<3 \times 10^{-4}$	>120	$<2 \times 10^{-4}$	0.2
Mercury(41)	5.5×10^{-1}	$\sim 9 \times 10^{-5}$	-	-	-	2×10^{-4}
Moon(42)	2.0×10^{-2}	4.0×10^{-5}	$<5 \times 10^{-8}$	>800	$<3 \times 10^{-6}$	2×10^{-3}

R = radius; J = angular momentum; $T^{(1)}$ = magnetic dipole moment; B = surface magnetic field (assumed dipolar). Subscript "E" refers to Earth. Subscript "obs" refers to observed values.

$[B/B_E]_{pred}$ = value predicted by Schuster's hypothesis = $(J/J_E)/(R/R_E)^3$

$T_E^{(1)} \simeq 8.0 \times 10^{22}\ A \cdot m^2$

$J_E \simeq 5.9 \times 10^{33}\ m^2 \cdot kg \cdot s^{-1}$

1.1.3 Schuster's hypothesis

Before continuing with a survey of dynamo theory, we must consider the possibility of intrinsic "rotational" magnetic fields. The conjecture that astrophysical fields are a necessary consequence of rotation has been advanced several times in the last 60 years *(Schuster, 1912; Wilson, 1923; Blackett, 1947, 1949; Milne, 1950; Papapetrou, 1950; Luchak, 1951; Moroz, 1967; Surdin, 1971; see the discussion in Rikitake, 1966a, p. 18)*. Interest has persisted despite Blackett's experimental demonstration that the hypothesis in its original form is false *(Blackett, 1952)*. The basic claim is that

$$J/T^{(1)} = \text{constant}$$

where J is the angular momentum of the body and $T^{(1)}$ the magnetic dipole moment. This relationship will be referred to as "Schuster's hypothesis" *(Warwick, 1971)*.

Using Schuster's hypothesis, *Blackett (1947)* was able to predict surface magnetic fields of 30 G for Jupiter and 3×10^6 G for white dwarfs - values which are not unreasonable in the light of recent observational evidence. *Warwick (1971)* has examined the relationship between angular momentum and dipole moment in astrophysical bodies, and has concluded that Schuster's hypothesis is not inconsistent with the observational data. He suggests that the Moon is the body most likely to provide a critical test of the hypothesis.

Since only surface fields are directly observable, it is necessary to write Schuster's hypothesis in a modified form. For a dipole magnetic field, the flux density at radius R and magnetic co-latitude θ is

$$B(R,\theta) = (\mu/4\pi R^3)\, T^{(1)}\{1 + 3\cos^2\theta\}^{1/2}$$

where μ is the magnetic permeability. The angular momentum of a body is given by

$$J = I\Omega$$

where Ω is the rotation frequency and I the moment of inertia. If it is assumed that $\mu = \mu_o$ = constant, and that all magnetic fields are evaluated at the same value of θ, Schuster's hypothesis can be rewritten

$$I\Omega/BR^3 = \text{constant}$$

where R is identified as the radius of the body in question. In this form, the hypothesis can be applied directly to the data available for planetary and stellar bodies.

Since Warwick's paper was written, new data have become available, particularly for the Moon *(Sharp, Russell, and Coleman, 1973)*. Furthermore, Warwick uses a value for the dipole moment of Mars which is considerably in excess of the estimate given by *Smith, et al. (1965; see Michaux, 1967, p. 51)*, even though both values are derived from the data provided by Mariner 4. (*Warwick's* value of the Mars/Earth dipole moment ratio is < 0.01, while that of *Smith, et al.* is < 0.0003.) The revised evidence to be used as a test

of Schuster's hypothesis is presented in the fourth column of Table 8. The observed values of the angular momentum/dipole moment ratio for planetary bodies span nearly three orders of magnitude. It would therefore appear that Schuster's hypothesis is incorrect. This conclusion is in agreement with Blackett's experimental result *(Blackett, 1952)*.

Values for the angular momentum/dipole moment ratio for stellar bodies are given in the last two columns of Table 7. In most cases (pulsars being the exception) a crude estimate of angular momentum has been obtained by making use of the relationship

$$I \propto MR^2 .$$

M is the mass of the body concerned, and the constant of proportionality depends on the internal density distribution. Assuming that the constant of proportionality in each case is the same as that for the Sun, we may write Schuster's hypothesis as

$$[M/M_{\odot}][\Omega/\Omega_{\odot}]/[B/B_{\odot}][R/R_{\odot}] \sim 1$$

where the subscript $\odot$ refers to solar values. From the last column of Table 7 it will be seen that when the stellar values of $[J/T^{(1)}]/[J_E/T_E^{(1)}]$ are added to the list of planetary values given in Table 8, the range spanned increases to nearly 5 orders of magnitude. (N.B. The subscript "E" refers to values for the Earth.)

The possibility remains, however, that Schuster's hypothesis is useful as an empirical relationship between the magnetic fields of bodies in which the physical conditions are not grossly different. *Warwick (1971)* has drawn attention to the close agreement between the values of $J/T^{(1)}$ for the Earth and Jupiter. *Scarf (1972)* has used Schuster's hypothesis to predict the magnetic field of Saturn from that observed for Jupiter, and concludes that the field he obtains (a polar field of 2 G, based on *Warwick's (1963)* value of 10 G for the field of Jupiter) is not inconsistent with radio observations. A further estimate could be made to predict the fields of Uranus and Neptune *(~2 G for a 14 G field on Jupiter, and ~7 G for a 50 G field on Jupiter; see Table 8)*. However, this approach carries with it all the dangers of "geophysical numerology" *(Jacobs, 1970a)*, and should be used with extreme caution.

It is interesting to note that Schuster's hypothesis is not consistent with both the "flux-conserving compression" hypothesis for the evolution of magnetic stars, and the conservation of angular momentum. Assuming both Schuster's hypothesis and the "flux-conservation" hypothesis to be valid, so that J/BR^3 and BR^2 are conserved, we find that J/R must be conserved as well. A stellar contraction would therefore have to be accompanied by a loss of angular momentum proportional to the change in radius. On the other

hand, if we assume both Schuster's hypothesis and the conservation of angular momentum to be valid, we see that BR^3 must be conserved in place of BR^2. It will be seen from the first column of Table 7 and the values of $R/R_{\odot}$ given in Table 6 that the spread of values of $[B/B_{\odot}][R/R_{\odot}]^2$ is much less than that for values of $[B/B_{\odot}][R/R_{\odot}]^3$.

FOOTNOTES: TABLES 1 TO 8

(1) Brecher & Blumenthal (1970); Parker (1970a); Kaplan & Pikel'ner (1970, p. 393).

(2) Ginzburg (1970); Kaplan & Pikel'ner (1970, p. 277).

(3) Pikel'ner (1968); Verschuur (1969); Parker (1969a, 1970a, 1971a, 1972); Jokipii & Parker (1969b); Jokipii & Lerche (1969); Piddington (1972a,b); Michel & Yahil (1973); Moffatt (1973).

(4) Mustel (1970); Woltjer (1972).

(5) Parker (1970a).

(6) Wilcox (1968); Ness (1968); Jokipii & Parker (1969a); Schatten (1971); Stenflo (1971); Sari & Fisk (1973).

(7) McElhinny & Evans (1968); Opdyke (1972); Roberts & Soward (1972).

(8) Lowes (1972); Roberts & Soward (1972).

(9) *The toroidal field was estimated by making use of expressions given by* Rochester (1960).

(10) Bullard & Gellman (1954); Hide (1966a); Braginskii (1971); Roberts & Soward (1972); Busse (1973b).

(11) Warwick (1963, 1967); Hide (1966b, 1969b, 1971a); Schatten & Ness (1971); Smoluchowski (1971, 1972).

(12) Smoluchowski (1971, 1972); Scarf (1972); Moffatt (1973).

(13) Moffatt (1973).

(14) Smith, et al. (1965); Michaux (1967).

(15) Koenig, et al. (1967); Van Allen, et al. (1967); Bridge, et al. (1967); Herman, et al. (1971).

(16) Banks, et al. (1970); Ness & Whang (1971).

(17) Helsley (1972b); Hide (1972); Coleman, Russell, Sharp, & Schubert (1972); Strangway & Sharpe (1973).

(18) Coleman, Schubert, Russell, & Sharp (1972); Sharp, Russell, & Coleman (1973).

(19) Parker (1970b); Kopecký (1970); Severny (1971); Stenflo (1972); Piddington (1972c).

(20) Severny, et al. (1970); Altschuler, et al. (1971); Wilcox & Gonzales (1971); Wilcox (1972); Svalgaard (1973); Patterson (1973); Schulz (1973).

(21) Weiss (1971a,b); Howard & Stenflo (1972).

(22) Parker (1970b); Weiss (1971a,b, 1972); Yoshimura (1972); Piddington (1972c); Moffatt (1973).

(23) Severny (1970, 1971).

(24) Severny (1970); Weiss (1971b).

(25) Ledoux & Renson (1966); Preston (1967a,b, 1971a,b); Landstreet (1970); Mestel (1967, 1971, 1972 - see Moffatt 1973); Spiegel (1972 - see Moffatt, 1973); Mestel & Takhar (1972); Raychaudhuri (1972).

(26) Angel & Landstreet (1970); Preston (1970); Trimble & Greenstein (1972).

(27) Kemp (1970); Kemp, et al. (1970); Angel & Landstreet (1971); Landstreet & Angel (1971).

(28) Ostriker & Gunn (1969); Gunn & Ostriker (1970, 1971); Cameron (1970); Hewish (1970); Ruderman (1972).

(29) Allen (1963).

(30) Allen (1963); Severny (1970, 1971).

(31) Allen (1963); Preston (1967a,b, 1971a); Eggen (1967); Mestel (1967).

(32) Ostriker (1968); Trimble & Greenstein (1972); Shipman (1972). *See note (26) for further references.*

(33) Ostriker & Bodenheimer (1968); Schwartz & Africk (1970); Ostriker (1971).

(34) Ostriker & Gunn (1969); Ruderman (1972); Greenstein (1972). *See note (28) for further references.*

(35) Ruderman (1972).

(36) Allen (1963). *See note (11) for further references.*

(37) Allen (1963). *See note (12) for further references.*

(38) Allen (1963); Moffatt (1973).

(39) Allen (1963); Dyce, et al. (1967); Shapiro (1967); Ash, et al. (1968); Melbourne, et al. (1968); Jurgens (1970); Anderson, et al. (1970). *See note (15) for further references.*

(40) Allen (1963); Smith, et al. (1965); Michaux (1967).

(41) Allen (1963); Dyce, et al. (1967).

(42) Allen (1963). *See note (18) for further references.*

1.2 The homogeneous dynamo problem

Theoretically, the homogeneous dynamo problem involves the solution of a highly complicated system of coupled partial differential equations. The best treatment of the derivation of these equations is that given by *P.H. Roberts (1967a)*. The outline given here is based on his approach.

The equations fall into four major groups, which will be considered separately.

a. The electrodynamic equations. These include Maxwell's equations, the constitutive relations among the various electric and magnetic fields, Ohm's Law, and the transformation relating the fields observed in one reference frame to those observed in another in relative motion.

b. The hydrodynamic equations. These include the equations of conservation of mass and conservation of momentum, and the constitutive equation for the total stress tensor.

c. The thermodynamic equations. These include the postulate of local thermodynamic equilibrium, the equation of heat conduction, and the constitutive law for the heat conduction vector. When combined with (a) and (b) above, these equations lead to a detailed description of energy flow within the system considered.

d. The boundary and initial conditions.

Over the years, the homogeneous dynamo problem has been attacked on three levels, corresponding to the first three groups of equations listed above. On the first level we have the *kinematic dynamo problem*, in which the fluid velocity is specified (independent of the magnetic field), and the electrodynamic equations are considered on their own. On the second level we have the *hydromagnetic dynamo problem*, in which the driving forces are specified (independent of the velocity and magnetic fields), and the electrodynamic and hydrodynamic equations are considered together. Finally, we have the *full hydromagnetic dynamo problem*, in which all three groups of equations are taken into consideration. In this thesis, we shall be concerned mainly with the first two levels of attack.

1.3 The kinematic dynamo problem

1.3.1 The dynamo equations

Let us first consider the kinematic dynamo problem. The form of Maxwell's equations appropriate to a moving conductor is:

$$\text{curl } \underset{\sim}{H} = \underset{\sim}{j} + \theta \underset{\sim}{u} + \partial \underset{\sim}{D}/\partial t \tag{1.1}$$

$$\text{curl } \underset{\sim}{E} = - \partial \underset{\sim}{B}/\partial t \tag{1.2}$$

$$\text{div } \underset{\sim}{B} = 0 \tag{1.3}$$

$$\text{div } \underset{\sim}{D} = \theta \tag{1.4}$$

where $\underset{\sim}{E}$ is the electric field, $\underset{\sim}{B}$ the magnetic flux density, $\underset{\sim}{D}$ the electric displacement, $\underset{\sim}{H}$ the magnetic field, $\underset{\sim}{j}$ the electric current density, θ the charge density, and $\underset{\sim}{u}$ the velocity of the medium. We shall assume that the constitutive equations for $\underset{\sim}{H}$ and $\underset{\sim}{D}$ are isotropic:

$$\underset{\sim}{H} = \underset{\sim}{B}/\mu \tag{1.5}$$

$$\underset{\sim}{D} = \epsilon \underset{\sim}{E} \tag{1.6}$$

where μ is the magnetic permeability and ϵ the permittivity. In this thesis we shall use SI units (see Appendix 1) and assume that $\mu = \mu_o = 4\pi \times 10^{-7}\ \text{H}\cdot\text{m}^{-1}$ everywhere. ϵ will be assumed constant for each material considered.

Ohm's Law, which should be valid in a frame moving locally with the medium whenever the particle density is

"sufficiently great" *(see the discussion in P.H. Roberts, 1967a, p. 9)*, is

$$\underset{\sim}{j}' = \sigma \underset{\sim}{E}' \tag{1.7}$$

where the conductivity σ is assumed isotropic. Primes are used to denote fields observed in the frame moving with the medium.

The equations relating the fields observed in two frames in relative motion are *(Landau and Lifshitz, 1951, §3-10; P.H. Roberts, 1967a, p. 10)*

$$\underset{\sim}{E}' = \gamma_u \left\{ \underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} \right\} + (1 - \gamma_u) \frac{\underset{\sim}{u} \cdot \underset{\sim}{E}}{u^2} \underset{\sim}{u} \tag{1.8}$$

$$\underset{\sim}{B}' = \gamma_u \left\{ \underset{\sim}{B} - \frac{\underset{\sim}{u} \times \underset{\sim}{E}}{c^2} \right\} + (1 - \gamma_u) \frac{\underset{\sim}{u} \cdot \underset{\sim}{B}}{u^2} \underset{\sim}{u} \tag{1.9}$$

where c is the speed of light, and

$$\gamma_u \equiv \left\{ 1 - \frac{u^2}{c^2} \right\}^{-1/2} \tag{1.10}$$

Substituting (1.5) and (1.6) into (1.1) and rearranging terms, we have

$$\underset{\sim}{j} = \frac{1}{\mu} \operatorname{curl} \underset{\sim}{B} - \theta \underset{\sim}{u} - \epsilon \, \partial \underset{\sim}{E} / \partial t$$

In the moving system, this equation reduces to

$$\underset{\sim}{j}' = \frac{1}{\mu} \operatorname{curl} \underset{\sim}{B}' - \epsilon \, \partial \underset{\sim}{E}' / \partial t$$

or, making use of (1.7)-(1.9),

$$\sigma\gamma_u(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B}) + \sigma(1-\gamma_u)\frac{\underset{\sim}{u}\cdot\underset{\sim}{E}}{u^2}\underset{\sim}{u}$$

$$= \frac{1}{\mu}\,\mathrm{curl}\left\{\gamma_u\underset{\sim}{B} - \gamma_u\frac{\underset{\sim}{u}\times\underset{\sim}{E}}{c^2} + (1-\gamma_u)\frac{\underset{\sim}{u}\cdot\underset{\sim}{B}}{u^2}\underset{\sim}{u}\right\}$$

$$-\epsilon\frac{\partial}{\partial t}\left\{\gamma_u(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B}) + (1-\gamma_u)\frac{\underset{\sim}{u}\cdot\underset{\sim}{E}}{u^2}\underset{\sim}{u}\right\}$$

Rearranging terms,

$$\begin{aligned}
&\gamma_u\left\{\frac{1}{\mu}\,\mathrm{curl}\,\underset{\sim}{B} - \sigma(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B})\right\} + \\
&+\left\{\frac{1}{\mu}\underset{\sim}{\nabla}\gamma_u\times\underset{\sim}{B} - \frac{1}{\mu}\gamma_u\underset{\sim}{\nabla}\times\left(\frac{\underset{\sim}{u}\times\underset{\sim}{E}}{c^2}\right) + \frac{1}{\mu}\underset{\sim}{\nabla}\times\left[(1-\gamma_u)\frac{\underset{\sim}{u}\cdot\underset{\sim}{B}}{u^2}\underset{\sim}{u}\right]\right. \\
&\qquad\qquad\left. - \sigma(1-\gamma_u)\frac{\underset{\sim}{u}\cdot\underset{\sim}{E}}{u^2}\underset{\sim}{u}\right\} + \\
&+\left\{\frac{1}{\mu}\underset{\sim}{\nabla}\gamma_u\times\frac{(\underset{\sim}{u}\times\underset{\sim}{E})}{c^2} + \epsilon\frac{\partial\gamma_u}{\partial t}\left[(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B}) - \frac{\underset{\sim}{u}\cdot\underset{\sim}{E}}{u^2}\underset{\sim}{u}\right]\right. \\
&\qquad\qquad\left. + (1-\gamma_u)\epsilon\frac{\partial}{\partial t}\left[\frac{\underset{\sim}{u}\cdot\underset{\sim}{E}}{u^2}\underset{\sim}{u}\right]\right\}
\end{aligned} \tag{1.11}$$

1.3.2 <u>The quasi-steady approximation</u>

In the *quasi-steady approximation (P.H. Roberts, 1967a, pp. 8-11)*, only the terms in the first set of braces in (1.11) are retained. Justification for this step is obtained by examining the scaling of the various terms. We assume that all the fields vary significantly on a length scale L and a time scale T, and replace space and time derivatives with $\frac{1}{L}$ and $\frac{1}{T}$ respectively. Further, we assume that

$$|\underset{\sim}{u}| \sim \mathcal{U} \sim L/T$$

and that

$$\epsilon \sim \epsilon_o = (\mu_o c^2)^{-1}$$

Then, from (1.2),

$$|\underset{\sim}{E}| \sim |\underset{\sim}{B}|\cdot(L/T)$$

and the three sets of braces in (1.11) have the ratio

$$(1 + R_m):(L/cT)^2(1 + R_m):(L/cT)^4(1)$$

where

$$R_m = \mu\sigma\mathcal{U}L = \frac{1}{\eta}\mathcal{U}L \qquad (1.12)$$

is a *magnetic Reynolds number* for the system, and

$$\eta = 1/\mu\sigma \qquad (1.13)$$

is the *magnetic diffusivity*. If $(L/cT)^2$ is small compared with unity - i.e. if the electromagnetic and velocity fields change very little in the time it takes light to cross the system - only the first set of terms in (1.11) need be retained, giving the equation

$$\eta \,\mathrm{curl}\, \underset{\sim}{B} = \underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} \qquad (1.14)$$

Similarly, the terms in equation (1.1) have the ratio

$$\{|\underset{\sim}{B}|/\mu L\}:\{|\underset{\sim}{j}|\}:\{(L/cT)^2[|\underset{\sim}{B}|/\mu L]\}:\{(L/cT)^2[|\underset{\sim}{B}|/\mu L]\}$$

so that, ignoring terms of order $(L/cT)^2$,

$$\underset{\sim}{j} = \frac{1}{\mu}\,\mathrm{curl}\,\underset{\sim}{B} = \sigma(\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}) \qquad (1.15)$$

Equation (1.15) gives the form of Ohm's Law in the laboratory frame appropriate to the quasi-steady approximation.

1.3.3 The magnetic induction equation

Taking the curl of equation (1.14) and making use of equation (1.2), we obtain the equation

$$\partial \underset{\sim}{B}/\partial t + \mathrm{curl}(\eta\, \mathrm{curl}\, \underset{\sim}{B}) = 0 \tag{1.16}$$

If η is independent of position, (1.16) becomes

$$\partial \underset{\sim}{B}/\partial t - \eta \nabla^2 \underset{\sim}{B} = 0 \tag{1.16'}$$

(1.16') is generally referred to as the *magnetic induction equation.*

Equations (1.3) and (1.16), together with the boundary and initial conditions and a specification of the velocity field $\underset{\sim}{u}$, provide a description of the kinematic dynamo problem in its simplest form. More generally, the kinematic dynamo problem involves finding a pair of fields $(\underset{\sim}{u}, \underset{\sim}{B})$ which satisfy (1.3), (1.6), the boundary and initial conditions, and certain additional conditions which are described in section 1.4.1.

It should be noted *(Krause, 1968a,b; G.O. Roberts, 1970a,b)* that equation (1.3) can be regarded as an initial condition. The divergence of (1.16) or (1.16') gives

$$\frac{\partial}{\partial t}[\mathrm{div}\, \underset{\sim}{B}] = 0$$

If $\mathrm{div}\, \underset{\sim}{B} = 0$ initially, this equation has the unique solution $\mathrm{div}\, \underset{\sim}{B} = 0$ for all time.

1.4 Solutions to the kinematic dynamo problem

1.4.1 Requirements on solutions

A "solution" to the kinematic dynamo problem consists of a pair of fields $(\underset{\sim}{u}, \underset{\sim}{B})$ which satisfy the following conditions:

a. $\underset{\sim}{u}$ is an *allowable flow (Gibson and Roberts, 1967; P.H. Roberts, 1967a, p. 66)*. In an *allowable flow*, the velocity gradients are everywhere finite, and the equation of continuity is satisfied everywhere without sources or sinks of mass. It must be possible to define non-singular distributions of body force and density which will generate the flow through the ordinary Navier-Stokes equation, but the flow is not required to satisfy the hydromagnetic Navier-Stokes equation *(see section 1.5.2)*.

b. $\underset{\sim}{B}$ is a field satisfying the magnetic induction equation (1.16), subject to the boundary and initial conditions.

c. If V is the volume occupied by the conducting fluid medium, the magnetic energy stored in V

$$\int_V \frac{B^2}{2\mu}\, d\underset{\sim}{r}$$

remains constant, grows with time, or oscillates about a mean value which itself remains constant or grows with time.

d. The total kinetic energy in the system

$$\int_V \frac{1}{2} \rho u^2 \, d\underset{\sim}{r}$$

is bounded *(Childress, 1968)*. ρ is the density of the conducting fluid.

In section 1.9.1, some of the mathematical implications of these equations will be examined in detail.

1.4.2 Anti-dynamo theorems

Numerous *anti-dynamo theorems* have been proved, placing further explicit restrictions on the nature of $\underset{\sim}{u}$ and $\underset{\sim}{B}$. These theorems, which are summarized in Tables 9 and 10, indicate that magnetic fields with a "simple" structure cannot, in general, be maintained by a dynamo process.

Cowling (1965) has listed three features which "...appear to be essential for any satisfactory dynamo theory". These features are summarized as follows by *Weiss (1971b)*:

> "...first, the velocity cannot be wholly irregular, for 'order does not arise spontaneously out of chaos'; secondly, two separate types of ordered motion should be present; and, thirdly, there should be an adequate dissipative mechanism."

These requirements should be kept in mind as we examine the various types of dynamo mechanism which have been proposed.

TABLE 9 - ANTI-DYNAMO THEOREMS FOR STATIONARY MAGNETIC FIELDS

Pairs of fields ($\mathbf{u}$, $\mathbf{B}$) with the following characteristics cannot be solutions to the kinematic dynamo problem, if it is required that $\partial\mathbf{B}/\partial t = 0$.

Velocity $\mathbf{u}$	Flux density $\mathbf{B}$	References
Arbitrary	Axisymmetric	Cowling (1933, 1957, 1968)
Arbitrary	Two-dimensional	Cowling (1957); Lortz (1968a)
Arbitrary	Poloidal	*see* Childress (1968)
Arbitrary	Toroidal	*see* Childress (1968)
Radial, vanishing at boundary	Arbitrary	Namikawa & Matsushita (1970)

N.B. A toroidal vector field is one which can be represented in the form curl $T\mathbf{r}$, where $\mathbf{r}$ is the position vector and T is a scalar field. A poloidal vector field is one which can be represented in the form curl curl $S\mathbf{r}$, where S is a scalar field. Toroidal and poloidal vector fields are solenoidal, by definition.

TABLE 10 - ANTI-DYNAMO THEOREMS FOR GENERAL MAGNETIC FIELDS

Pairs of fields ($\underset{\sim}{u}$, $\underset{\sim}{B}$) with the following characteristics cannot be solutions to the kinematic dynamo problem.

Velocity $\underset{\sim}{u}$	Flux density $\underset{\sim}{B}$	References
Toroidal (in a sphere)	Arbitrary	Bullard & Gellman (1954); Cowling (1957).
Axisymmetric	Axisymmetric	Backus & Chandrasekhar (1956); Backus (1957); Cowling (1957); Braginskii (1964a).
Pure sin $m\phi$ or cos $m\phi$ dependence about axis of $\underset{\sim}{B}$ "symmetry"	Nearly axisymmetric	Braginskii (1964b); P.H. Roberts (1967b).
"Slow" (magnetic Reynolds number for mean flow < 1)	Arbitrary	Childress (1968); P.H. Roberts (1967a,b).
"Insufficient rates of strain" ($\partial u_i/\partial x_j$ too small)	Arbitrary	Backus (1958); Childress (1968).
Arbitrary, in a bounded, simply connected, perfectly conducting medium surrounded by a nonconductor.	Extending into the nonconducting medium	Bondi & Gold (1950); Leorat (1969).

1.4.3 Existence of solutions

It has been shown that solutions to the kinematic dynamo problem do exist. In the decade and a half since the first existence proofs were published *(Herzenberg, 1958; Backus, 1958)*, a large number of successful dynamo models have been developed. Broadly speaking, these models fall into five main classes, according to the method used in solving the induction equation. These five classes are summarized in sections 1.4.4-1.4.8.

1.4.4 "Exact" models

Exact analytical solutions of the kinematic dynamo problem are not common. The most important example in this class is the "helical" dynamo of *Lortz (1968b)*, which operates in an unbounded conductor. The streamlines of $\underset{\sim}{u}$ are concentric helices with constant cross-sectional area.

1.4.5 Spherical harmonic expansion models

Expansion of $\underset{\sim}{u}$ and $\underset{\sim}{B}$ in spherical harmonics materially simplifies the equations governing kinematic dynamo action in a sphere. Unfortunately, because of the "interaction" term $\underset{\sim}{u} \times \underset{\sim}{B}$ in the induction equation, $\underset{\sim}{B}$ must generally be represented by an infinite set of harmonics even when $\underset{\sim}{u}$ has a simple form, so that trunca-

tion difficulties arise. The problem was studied by a number of workers in the late 1940's and early 1950's for the case of a steady magnetic field, $\partial \underset{\sim}{B}/\partial t = 0$ *(Elsasser, 1946a,b, 1947; Takeuchi and Shimazu*, 1952a,b, 1953, 1954; Bullard and Gellman, 1954)*. However, no convincing evidence of dynamo action was obtained. Since that time, convergent models have been developed by *Gubbins (1972, 1973)* and by *Roberts and Kumar (1972)*.

1.4.6 "Sporadic" models

In *sporadic models* the effects of the terms curl $(\underset{\sim}{u} \times \underset{\sim}{B})$ and $\eta\nabla^2\underset{\sim}{B}$ in the induction equation (1.16') are separated by choosing a "sporadic" velocity field *(Bade, 1954; Parker, 1955)*. The dynamo alternates between periods of motion sufficiently rapid and short-lived for diffusion to be neglected, and periods in which the motion is stopped to allow for the "simplification" of spatially complex $\underset{\sim}{B}$-fields by diffusive decay of the higher harmonics. Successful dynamo models have been developed by *Backus (1958)* and *Tverskoy (1966)*, using motions in a sphere. *(See also P.H. Roberts, 1971a.)*

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* *Backus (1957)* points out that Takeuchi and Shimazu have set too many boundary conditions. It is therefore highly unlikely that their numerical "evidence" for steady dynamo action is meaningful.

1.4.7 Asymptotic models

Asymptotic dynamo models are generally characterized by the presence of two different "length" scales. These models fall into three major groups.

a. Herzenberg-type dynamos. In models of this type, $\mathbf{u}$ varies on a small length scale ℓ, while $\mathbf{B}$ has a component varying on a large length scale L. The magnetic Reynolds number R_m based on the smaller length scale is allowed to approach infinity while the ratio ℓ/L goes to zero and the product $R_m(\ell/L)^3$ remains finite. *(Herzenberg, 1958; P.H. Roberts, 1967b, pp. 95-104; Gibson, 1968a,b, 1969; Kropachev, 1964, 1965, 1966; Gailitis, 1970; P.H. Roberts, 1971a.)*

b. Periodic dynamos. In models of this type, $\mathbf{u}$ is periodic with a short wavelength ℓ, while $\mathbf{B}$ is doubly periodic with both the wavelength ℓ and a much larger wavelength L. The magnetic Reynolds number R_m based on the shorter wavelength is allowed to approach zero with ℓ/L, while the product $R_m^2(L/\ell)$ remains finite. *(Childress, 1967a,b,c, 1968, 1969, 1970; G.O. Roberts, 1969, 1970a,b, 1972a).*

G.O. Roberts (1970a,b) has shown that "nearly all" periodic motions in an unbounded conductor will lead to dynamo action. Furthermore, *Childress (1967b, 1968, 1970)* has shown that a periodic motion giving dynamo

action in an unbounded conductor retains this property when fitted into a finite spherical volume by means of a "cut-off" function.

G.O. Roberts (1973; see P.H. Roberts, 1971b) and *Gubbins (1972, 1973)* have carried out numerical studies of "cellular" dynamos in a sphere for axisymmetric velocity fields $\underset{\sim}{u}$. While the motions considered are not "periodic" in the sense defined above, these models are included here for the sake of illustration.

c. Nearly axisymmetric dynamos. In models of this type, $\underset{\sim}{u}$ and $\underset{\sim}{B}$ are required to tend toward axial symmetry as the magnetic Reynolds number approaches infinity. *(Braginskii, 1964a,b,c; Tough, 1967; Tough and Gibson, 1969; Soward, 1971a, 1972a; P.H. Roberts, 1967b, pp. 105-127; P.H. Roberts, 1971a.)*

The asymptotic limit used in *nearly axisymmetric dynamos* can be interpreted in terms of two "length" scales if an "azimuthal length scale" L is defined by means of the ratio

$$\frac{\ell}{L} \sim \frac{|\underset{\sim}{1}_\phi \cdot \underset{\sim}{\nabla} u|}{|\underset{\sim}{\nabla} u|} \sim \frac{|\underset{\sim}{1}_\phi \cdot \underset{\sim}{\nabla} B|}{|\underset{\sim}{\nabla} B|}$$

ℓ is the length scale of variation of $\underset{\sim}{u}$ and $\underset{\sim}{B}$ in meridian planes, $\underset{\sim}{1}_\phi$ the unit vector in the azimuthal direction, and u and B the magnitudes of $\underset{\sim}{u}$ and $\underset{\sim}{B}$. The magnetic Reynolds number R_m based on ℓ is

allowed to approach infinity while ℓ/L goes to zero and the product $R_m(\ell/L)^2$ remains finite.

Soward (1972a) has pointed out that, for the case of a steady dynamo with closed streamlines, a better interpretation of the asymptotic limit is obtained by requiring that the integral

$$\frac{1}{\mathcal{U}} \oint_{C(\underset{\sim}{x})} \frac{dx}{|\underset{\sim}{u}|} (\underset{\sim}{u} \cdot \mathrm{curl}\, \underset{\sim}{u})$$

approach zero as R_m^{-1} when R_m goes to infinity. Here $\mathcal{U}$ is a scaling amplitude for the velocity field, $C(\underset{\sim}{x})$ the contour of a streamline, and $\underset{\sim}{x}$ the position vector of points on the streamline. The integral is, in a sense, an average of $\underset{\sim}{u} \cdot \mathrm{curl}\, \underset{\sim}{u}$ for the flow, and may be considered as a measure of the *helicity* - a quantity which is very important in turbulent dynamo models (see section 1.4.8).

1.4.8 Mean field models

In *mean field dynamo models*, $\underset{\sim}{u}$ and $\underset{\sim}{B}$ are each represented as the sum of a statistical average and a fluctuating part. The average fields $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$ are assumed to vary on a length scale L , while the fluctuating fields $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ (with zero statistical average) are assumed to vary on a length scale ℓ . In this sense the problem is related to the *two-scale* approach considered in the last section.

The statistical average of Ohm's Law for a moving medium (i.e. the average of equation 1.15) is

$$\bar{\underset{\sim}{J}} = \sigma(\bar{\underset{\sim}{E}} + \bar{\underset{\sim}{u}} \times \bar{\underset{\sim}{B}} + \overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}) \tag{1.17}$$

This equation contains a "new" electromotive force $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$. If this e.m.f. can be represented as a functional of the mean fields $\bar{\underset{\sim}{u}}$ and $\bar{\underset{\sim}{B}}$, the mean field kinematic dynamo problem becomes closed. Considerable attention has been focussed on the derivation of simple representations for $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$. *Parker (1955)* drew attention to the possibility that

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'} = \underset{\approx}{\alpha} \cdot \bar{\underset{\sim}{B}} \tag{1.18}$$

Steenbeck and Krause (1966) have christened this term the α-*effect*. Several successful α-effect dynamos have been studied, both for the case $\bar{\underset{\sim}{u}} = 0$ *(Krause and Steenbeck, 1967; Steenbeck and Krause, 1966, 1967; Moffatt, 1970a; Leorat, 1969)* and for the case $\bar{\underset{\sim}{u}} \neq 0$ *(Parker, 1955, 1970a,b,c, 1971a-f; Krause and Steenbeck, 1965; Steenbeck and Krause, 1966, 1967, 1969a,b; Lerche and Parker, 1971, 1972)*. Models in which $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ has a more complicated dependence on $\bar{\underset{\sim}{u}}$ and $\bar{\underset{\sim}{B}}$ than that given by (1.18) have also been studied *(Steenbeck, Krause, and Rädler, 1966; Rädler, 1966, 1968a,b, 1969a,b, 1970; Krause and Rädler, 1971; P.H. Roberts, 1971a)*.

It should be noted that, to a first approximation, a fully isotropic turbulent motion cannot support dynamo action *(Gilliland and Aldridge, 1973; Krause and Roberts,*

1973; see section 3.2 below). In order for dynamo action to occur, the turbulence must have *helicity (Moffatt, 1969)* - i.e.

$$\overline{\underset{\sim}{u}' \cdot \mathrm{curl}\ \underset{\sim}{u}'} \neq 0$$

- or be anisotropic with a preferred direction *(see, for example, Krause and Rädler, 1971; P.H. Roberts, 1971a).*

1.5 The hydromagnetic dynamo problem

Let us now turn from the kinematic dynamo problem to the more complicated hydromagnetic dynamo problem. The magnetic flux density $\underset{\sim}{B}$ must still satisfy the induction equation (1.16), subject to boundary and initial conditions, but now the velocity field $\underset{\sim}{u}$ must itself be derived from a specified body force distribution by solving the hydrodynamic equations subject to appropriate boundary and initial conditions.

1.5.1 The equation of mass conservation

The first of the hydrodynamic equations to be considered is the equation of *conservation of mass*

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho = -\rho \operatorname{div} \underset{\sim}{u} \qquad (1.19)$$

In equation (1.19),

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla} \qquad (1.20)$$

is the Lagrangian time derivative *(i.e. the "material" derivative following the flow)*, and ρ is the density of the conducting fluid medium. For a truly *incompressible* fluid, the left hand side of (1.19) vanishes identically, giving

$$\operatorname{div} \underset{\sim}{u} = 0 \qquad (1.21)$$

If $\partial\rho/\partial t$ is identically zero but density gradients are present, (1.19) reduces to

$$\operatorname{div} \underset{\sim}{u} = -\frac{1}{\rho} \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \rho \tag{1.21'}$$

In the full hydromagnetic dynamo problem, $D\rho/Dt$ must be evaluated from the *thermodynamic* equations.

1.5.2 The Navier-Stokes equation of hydromagnetics

The second hydrodynamic equation of interest is the equation of *conservation of momentum*

$$\rho \frac{Du_i}{Dt} = \rho \left\{ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right\} = \frac{\partial p_{ij}}{\partial x_j} + F_i \tag{1.22}$$

where p_{ij} is the *total stress tensor*, $\underset{\sim}{F}$ the *applied body force per unit volume*, and x_i a component of the position vector $\underset{\sim}{r}$. The constitutive equation for p_{ij} can be assumed to have the form *(P.H. Roberts, 1967a, p. 17)*

$$p_{ij} = -p\,\delta_{ij} + \Pi_{ij} + m_{ij} \tag{1.23}$$

where p is the kinetic pressure, Π_{ij} the viscous stress tensor, and m_{ij} the electromagnetic stress tensor. For a Newtonian fluid,

$$\Pi_{ij} = \rho\left\{\zeta - \frac{2}{3}\nu\right\}\frac{\partial u_k}{\partial x_k}\delta_{ij} + \rho\nu\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\} \tag{1.24}$$

where ν is the kinematic (shear) viscosity and ζ the kinematic bulk viscosity. The electromagnetic stress tensor is defined by

$$\frac{\partial}{\partial x_j} m_{ij} = \Theta E_i + (\underset{\sim}{j} \times \underset{\sim}{B})_i$$

However, with the scaling used in deriving the induction equation (1.16), $|\theta \underset{\sim}{E}|$ is of order $(L/cT)^2$ compared with $|\underset{\sim}{j} \times \underset{\sim}{B}|$, so that, in the *quasi-steady approximation*,

$$\frac{\partial}{\partial x_j} m_{ij} = (\underset{\sim}{j} \times \underset{\sim}{B})_i \tag{1.25}$$

From (1.25) and (1.15) it follows that the form of the (electro)magnetic stress tensor is

$$m_{ij} = \frac{1}{\mu}(B_i B_j - \frac{1}{2}B^2 \delta_{ij}) \tag{1.26}$$

(see P.H. Roberts, 1967a, p. 11). Finally, from (1.23), (1.24), and (1.26),

$$\begin{aligned} P_{ij} = &- \left\{ p + \frac{1}{2\mu} B^2 - \rho(\zeta - \tfrac{2}{3}\nu) \operatorname{div} \underset{\sim}{u} \right\} \delta_{ij} \\ &+ \rho\nu \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\} + \frac{1}{\mu} B_i B_j \end{aligned} \tag{1.27}$$

Substituting (1.27) into (1.22), we obtain the *Navier-Stokes equation of hydromagnetics.* This equation has two alternative forms *(P.H. Roberts, 1967a, p. 17)*

$$\begin{aligned} \rho \frac{Du_i}{Dt} = &- \frac{\partial}{\partial x_i} \left\{ p + \frac{B^2}{2\mu} - \rho(\zeta - \tfrac{2}{3}\nu) \operatorname{div} \underset{\sim}{u} \right\} \\ &+ \frac{\partial}{\partial x_j} \left\{ \rho\nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\} + \frac{1}{\mu} B_j \frac{\partial B_i}{\partial x_j} + F_i \end{aligned} \tag{1.28}$$

$$\begin{aligned} \rho \frac{Du_i}{Dt} = &- \frac{\partial}{\partial x_i} \left\{ p - \rho(\zeta - \tfrac{2}{3}\nu) \operatorname{div} \underset{\sim}{u} \right\} + F_i \\ &+ \frac{\partial}{\partial x_j} \left\{ \rho\nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\} + \frac{1}{\mu} \left\{ (\operatorname{curl} \underset{\sim}{B}) \times \underset{\sim}{B} \right\}_i \end{aligned} \tag{1.29}$$

depending on the form in which the *Lorentz force* $\underset{\sim}{j} \times \underset{\sim}{B}$ is written. For the *incompressible* case when (1.21) is valid, (1.28) and (1.29) reduce to

$$Du/Dt = -\frac{1}{\rho}\nabla\left\{p + \frac{B^2}{2\mu}\right\} + \nu\nabla^2 u + \frac{1}{\rho\mu}B\cdot\nabla B + \frac{1}{\rho}F \tag{1.30}$$

$$Du/Dt = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u + \frac{1}{\rho\mu}(\mathrm{curl}\, B)\times B + \frac{1}{\rho}F \tag{1.31}$$

assuming that ν is a constant. Alternatively, if (1.21') is the appropriate form of the mass conservation equation, and ν is a constant, the terms involving density gradients in (1.28) and (1.29) can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial x_i}\left\{\rho(\zeta - \tfrac{2}{3}\nu)\,\mathrm{div}\, u\right\} + \frac{\partial}{\partial x_j}\left\{\rho\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} \\ &\quad = \left\{-\nabla\left[(\zeta + \tfrac{1}{3}\nu)u\cdot\nabla\rho\right] + \rho\nu\nabla^2 u \right. \\ &\qquad \left. + \nu\left[\tfrac{1}{\rho}(u\cdot\nabla\rho)\nabla\rho + \nabla\rho\cdot\nabla u + \nabla u\cdot\nabla\rho\right]\right\}_i \end{aligned} \tag{1.32}$$

1.5.3 The hydromagnetic equations in a rotating frame

In most cases of interest, we must deal with a *rotating* conducting fluid. It is therefore useful to transform the hydromagnetic dynamo equations to a rotating frame of reference. The velocity u in the non-rotating frame can be expressed as the sum of a uniform rotation with angular velocity Ω, and a velocity u_{rot} relative to the rotating frame.

$$u = u_{rot} + \Omega \times r \tag{1.33}$$

The Lagrangian time derivative (1.20) transforms as

$$\frac{D}{Dt}G = \left(\frac{D}{Dt}\right)_{rot} G + \Omega\times G \tag{1.34}$$

where

$$\left(\frac{D}{Dt}\right)_{rot} \underset{\sim}{G} \equiv \left(\frac{\partial}{\partial t}\right)_{rot} \underset{\sim}{G} + \underset{\sim}{u}_{rot} \cdot \underset{\sim}{\nabla} \underset{\sim}{G} \qquad (1.35)$$

and $\underset{\sim}{G}$ is any vector quantity. It follows that the Eulerian time derivative $\partial/\partial t$ transforms as

$$\frac{\partial}{\partial t} \underset{\sim}{G} = \left(\frac{\partial}{\partial t}\right)_{rot} \underset{\sim}{G} + \underset{\sim}{\nabla} \times \{(\underset{\sim}{\Omega} \times \underset{\sim}{r}) \times \underset{\sim}{G}\} - (\underset{\sim}{\nabla} \cdot \underset{\sim}{G})(\underset{\sim}{\Omega} \times \underset{\sim}{r}) \qquad (1.36)$$

Applying (1.36) to the induction equation (1.16'), making use of (1.33) we find that the form of the equation is invariant.

$$\left(\frac{\partial}{\partial t}\right)_{rot} \underset{\sim}{B} + \mathrm{curl}\{(\underset{\sim}{\Omega} \times \underset{\sim}{r}) \times \underset{\sim}{B}\} - \eta \nabla^2 \underset{\sim}{B}$$
$$= \mathrm{curl}\{\underset{\sim}{u}_{rot} \times \underset{\sim}{B} + (\underset{\sim}{\Omega} \times \underset{\sim}{r}) \times \underset{\sim}{B}\}$$

so that

$$\left\{\left(\frac{\partial}{\partial t}\right)_{rot} - \eta \nabla^2\right\} \underset{\sim}{B} = \mathrm{curl}\{\underset{\sim}{u}_{rot} \times \underset{\sim}{B}\} \qquad (1.37)$$

Strictly speaking this invariance holds only if the speed of absolute motion $|\underset{\sim}{u}|$ is much less than the speed of light. *(See Trocheris, 1949; Backus, 1958; Acheson and Hide, 1973.)* *Backus (1958)* points out that the corresponding result for the electric field is false. *(See Backus, 1956 for the effect on* $\underset{\sim}{E}$ *of a superposed rigid rotation.)*

Applying (1.34) to (1.33), we have

$$D\underset{\sim}{u}/Dt = (D/Dt)_{rot} \underset{\sim}{u}_{rot} + 2\underset{\sim}{\Omega} \times \underset{\sim}{u}_{rot} + (\partial \underset{\sim}{\Omega}/\partial t) \times \underset{\sim}{r}$$
$$- \tfrac{1}{2} \underset{\sim}{\nabla} \{|\underset{\sim}{\Omega} \times \underset{\sim}{r}|^2\}$$

or

$$D\underset{\sim}{u}/Dt = (D/Dt)_{rot}\,\underset{\sim}{u}_{rot} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}_{rot} + (\partial\underset{\sim}{\Omega}/\partial t)_{rot}\times\underset{\sim}{r} - \underset{\sim}{\nabla}\{\tfrac{1}{2}|\underset{\sim}{\Omega}\times\underset{\sim}{r}|^2\} \tag{1.38}$$

$$= (D/Dt)_{rot}\,\underset{\sim}{u}_{rot} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}_{rot} + (\partial\underset{\sim}{\Omega}/\partial t)_{rot}\times\underset{\sim}{r} + \underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{r}) \tag{1.38'}$$

Also, it follows from (1.33) that

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = \underset{\sim}{\nabla}\cdot\underset{\sim}{u}_{rot} \tag{1.39}$$

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_j}(\underset{\sim}{u}_{rot})_i + \frac{\partial}{\partial x_i}(\underset{\sim}{u}_{rot})_j \tag{1.40}$$

while for any scalar field a

$$Da/Dt = (Da/Dt)_{rot} \tag{1.41}$$

$$\partial a/\partial t = (\partial a/\partial t)_{rot} - (\underset{\sim}{\Omega}\times\underset{\sim}{r})\cdot\underset{\sim}{\nabla}a \tag{1.42}$$

The equation of conservation of mass in the rotating frame thus has the same form as (1.19)

$$(D\rho/Dt)_{rot} = -\rho\,\underset{\sim}{\nabla}\cdot\underset{\sim}{u}_{rot} \tag{1.43}$$

while the Navier-Stokes equation (1.28, 1.29) becomes

$$\begin{aligned}\rho\Big\{\Big(\frac{D}{Dt}\Big)_{rot}\underset{\sim}{u}_{rot} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}_{rot} + (\partial\underset{\sim}{\Omega}/\partial t)_{rot}\times\underset{\sim}{r}\Big\}_i \\ = -\frac{\partial}{\partial x_i}\Big\{p + \frac{B^2}{2\mu} - \rho(\zeta - \tfrac{2}{3}\nu)\underset{\sim}{\nabla}\cdot\underset{\sim}{u}_{rot} - \tfrac{1}{2}\rho|\underset{\sim}{\Omega}\times\underset{\sim}{r}|^2\Big\} \\ - \tfrac{1}{2}(\underset{\sim}{\nabla}\rho)_i|\underset{\sim}{\Omega}\times\underset{\sim}{r}|^2 + \frac{\partial}{\partial x_j}\Big\{\rho\nu\Big(\frac{\partial}{\partial x_j}[\underset{\sim}{u}_{rot}]_i + \frac{\partial}{\partial x_i}[\underset{\sim}{u}_{rot}]_j\Big)\Big\} \\ + \frac{1}{\mu}B_j\frac{\partial B_i}{\partial x_j} + F_i\end{aligned} \tag{1.44}$$

or

$$\rho\left\{\left(\frac{D}{Dt}\right)_{rot}\underset{\sim}{u}_{rot} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}_{rot} + (\partial\underset{\sim}{\Omega}/\partial t)_{rot}\times\underset{\sim}{r}\right\}_i$$

$$= -\frac{\partial}{\partial x_i}\left\{p - \rho(\zeta - \tfrac{2}{3}\nu)\underset{\sim}{\nabla}\cdot\underset{\sim}{u}_{rot}\right\} - \rho\underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{r})$$

$$+ \frac{\partial}{\partial x_j}\left\{\rho\nu\left(\frac{\partial}{\partial x_j}[\underset{\sim}{u}_{rot}]_i + \frac{\partial}{\partial x_i}[\underset{\sim}{u}_{rot}]_j\right)\right\}$$

$$+ \frac{1}{\mu}\left\{(\text{curl}\,\underset{\sim}{B})\times\underset{\sim}{B}\right\}_i + F_i \qquad (1.44')$$

In this thesis we shall restrict attention to the hydromagnetic dynamo equations in the rotating frame, and drop the subscript "rot". The hydromagnetic dynamo problem is then specified by the equations

$$\left\{\partial/\partial t - \eta\nabla^2\right\}\underset{\sim}{B} = \text{curl}\,(\underset{\sim}{u}\times\underset{\sim}{B}) \qquad (1.45)$$

$$D\rho/Dt = -\rho\,\underset{\sim}{\nabla}\cdot\underset{\sim}{u} \qquad (1.46)$$

$$D\underset{\sim}{u}/Dt + 2\underset{\sim}{\Omega}\times\underset{\sim}{u} = -\frac{1}{\rho}\underset{\sim}{\nabla}P - \frac{1}{2\rho}|\underset{\sim}{\Omega}\times\underset{\sim}{r}|^2\,\underset{\sim}{\nabla}\rho - (\partial\underset{\sim}{\Omega}/\partial t)\times\underset{\sim}{r}$$

$$+ \nu\left\{\nabla^2\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right\} + \frac{\nu}{\rho}\left\{\underset{\sim}{\nabla}\rho\cdot\underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho\right\}$$

$$+ \frac{1}{\rho\mu}\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\underset{\sim}{B} + \frac{1}{\rho}\underset{\sim}{F} \qquad (1.47)$$

$$= -\frac{1}{\rho}\underset{\sim}{\nabla}\left\{p - \rho(\zeta - \tfrac{2}{3}\nu)\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right\} - \underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{r})$$

$$+ \nu\left\{\nabla^2\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right\} + \frac{\nu}{\rho}\left\{\underset{\sim}{\nabla}\rho\cdot\underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho\right\}$$

$$- (\partial\underset{\sim}{\Omega}/\partial t)\times\underset{\sim}{r} + \frac{1}{\rho\mu}(\underset{\sim}{\nabla}\times\underset{\sim}{B})\times\underset{\sim}{B}$$

$$+ \frac{1}{\rho}\underset{\sim}{F} \qquad (1.47')$$

where ν has been assumed constant, and

$$\mathbf{P} \equiv p + \frac{B^2}{2\mu} - \rho(\zeta - \tfrac{2}{3}\nu)\nabla\cdot\mathbf{u} - \tfrac{1}{2}\rho|\boldsymbol{\Omega}\times\mathbf{r}|^2 \tag{1.48}$$

In the simple *incompressible* case, (1.45)-(1.48) reduce to

$$\{D/Dt - \eta\nabla^2\}\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} \tag{1.45a}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{1.46a}$$

$$D\mathbf{u}/Dt + 2\boldsymbol{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla\mathbf{P} - (\partial\boldsymbol{\Omega}/\partial t)\times\mathbf{r} + \nu\nabla^2\mathbf{u} + \frac{1}{\rho\mu}\mathbf{B}\cdot\nabla\mathbf{B} + \frac{1}{\rho}\mathbf{F} \tag{1.47a}$$

$$= -\frac{1}{\rho}\nabla p - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) - (\partial\boldsymbol{\Omega}/\partial t)\times\mathbf{r} + \nu\nabla^2\mathbf{u} + \frac{1}{\rho\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \frac{1}{\rho}\mathbf{F} \tag{1.47'a}$$

$$\mathbf{P} \equiv p + \frac{B^2}{2\mu} - \tfrac{1}{2}\rho|\boldsymbol{\Omega}\times\mathbf{r}|^2 \tag{1.48a}$$

This system must be solved for $\mathbf{u}$, $\mathbf{B}$, and p subject to the boundary and initial conditions when $\mathbf{F}$ and ρ are given.

If density gradients are present, but the density distribution does not vary with time, (1.45)-(1.48) reduce to

$$\{D/Dt - \eta\nabla^2\}\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \frac{1}{\rho}(\mathbf{u}\cdot\nabla\rho)\mathbf{B} \tag{1.45b}$$

$$\nabla\cdot\mathbf{u} = -\frac{1}{\rho}\mathbf{u}\cdot\nabla\rho \tag{1.46b}$$

$$D\underset{\sim}{u}/Dt + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} = -\frac{1}{\rho}\underset{\sim}{\nabla}P' - (\partial\underset{\sim}{\Omega}/\partial t) \times \underset{\sim}{r} - \frac{1}{2\rho}|\underset{\sim}{\Omega} \times \underset{\sim}{r}|^2 \underset{\sim}{\nabla}\rho$$
$$+ \nu\nabla^2\underset{\sim}{u} + \frac{\nu}{\rho}\left\{\frac{1}{\rho}(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho)\underset{\sim}{\nabla}\rho + \underset{\sim}{\nabla}\rho\cdot\underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho\right\}$$
$$+ \frac{1}{\rho\mu}\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\underset{\sim}{B} + \frac{1}{\rho}\underset{\sim}{F} \qquad (1.47b)$$

$$= -\frac{1}{\rho}\underset{\sim}{\nabla}\left\{p + (\zeta + \tfrac{1}{3}\nu)\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho\right\} - \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r})$$
$$+ \nu\nabla^2\underset{\sim}{u} + \frac{\nu}{\rho}\left\{\frac{1}{\rho}(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho)\underset{\sim}{\nabla}\rho + \underset{\sim}{\nabla}\rho\cdot\underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho\right\}$$
$$- (\partial\underset{\sim}{\Omega}/\partial t) \times \underset{\sim}{r} + \frac{1}{\rho\mu}(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B}$$
$$+ \frac{1}{\rho}\underset{\sim}{F} \qquad (1.47'b)$$

$$P' \equiv p + \frac{B^2}{2\mu} + (\zeta + \tfrac{1}{3}\nu)\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\rho - \tfrac{1}{2}\rho|\underset{\sim}{\Omega} \times \underset{\sim}{r}|^2 \qquad (1.48b)$$

This system may be solved for $\underset{\sim}{u}$, $\underset{\sim}{B}$, and ρ , subject to the boundary and initial conditions, when $\underset{\sim}{F}$ and p are given, <u>or</u> for $\underset{\sim}{u}$, $\underset{\sim}{B}$, and p , subject to the same conditions, when $\underset{\sim}{F}$ and ρ are specified.

1.6 The full hydromagnetic dynamo problem - thermodynamic equations

In most cases of interest, the hydromagnetic dynamo problem is incomplete without consideration of the *thermodynamic equations*. Two scalar equations are added to the system (1.45)-(1.47), giving nine scalar equations in the nine unknowns [$\underset{\sim}{u}$, $\underset{\sim}{B}$, p , ρ , T], where T is the temperature. The additional equations can be written in the form *(P.H. Roberts, 1967a, pp. 12-16, 18-22)*

$$\rho c_p \, DT/Dt - \bar{\alpha} T \, Dp/Dt = \underset{\sim}{\nabla} \cdot (\lambda \underset{\sim}{\nabla} T) + \Pi_{ij} \frac{\partial u_i}{\partial x_j} + \frac{1}{\sigma} j^2 + \epsilon_s \tag{1.49}$$

$$D\rho/Dt = \left\{ \frac{1}{a^2} + \frac{\bar{\alpha}^2 T}{c_p} \right\} Dp/Dt - \bar{\alpha} \rho \, DT/Dt \tag{1.50}$$

when the system is assumed to be in local thermodynamic equilibrium, and a Fourier Law is assumed to hold for the heat conduction vector. Here λ is the thermal conductivity, $\bar{\alpha}$ the volume expansion coefficient, c_p the specific heat per unit mass at constant pressure, a the adiabatic speed of sound, and ϵ_s the rate, per unit volume, at which sources of heat provide energy within the fluid. In general, λ , $\bar{\alpha}$, and a are all functions of density, pressure, and temperature. It should be noted that effects due to variations of chemical composition have been neglected in equation (1.50). This equation is sometimes

replaced by an equation of state, of the form

$$\rho = \rho(T,\ p;\ \text{chemical composition}) \qquad (1.50')$$

(see, for example, Hide, 1969b).

When velocities in the system are small compared with the speed of sound, (1.49) and (1.50) reduce to

$$DT/Dt = \nabla(\lambda \nabla T) + \Pi_{ij}\frac{\partial u_i}{\partial x_j} + \frac{1}{\sigma} j^2 + \epsilon_s \qquad (1.51)$$

$$D\rho/Dt = \bar{\alpha}\rho\, DT/Dt \qquad (1.52)$$

(P.H. Roberts, 1967a, p. 16). Further simplifications may be made once the scaling of the various terms in the equations is known *(see section 6.1)*. The full hydromagnetic dynamo problem requires equations (1.45)-(1.48) and (1.51)-(1.52) to be solved for [$\underset{\sim}{u}$, $\underset{\sim}{B}$, p , ρ , T] subject to the boundary and initial conditions, once the independent body forces and sources of heat are known.

1.7 The full hydromagnetic dynamo problem - boundary conditions

The boundary conditions which must be satisfied by solutions to the full hydromagnetic dynamo problem have been given in some detail by *P.H. Roberts (1967a, pp. 22-28)*. These conditions fall into three major groups, which will be discussed separately in the next three sections.

1.7.1 Electromagnetic boundary conditions

The fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$ must satisfy a total of four independent scalar conditions at any surface of discontinuity of σ , ϵ , or μ . These conditions can be written in a number of equivalent forms. Caution must be exercised, however, if the magnetic diffusivity $\eta = 0$ in any region *(see P.H. Roberts, 1967a, pp. 24-26)*. In this thesis, we shall deal exclusively with the case $\eta \neq 0$.

Perhaps the simplest form of the electromagnetic boundary conditions when $\eta \neq 0$ is

$$\langle \underset{\sim}{n} \times \underset{\sim}{B} \rangle = 0 \qquad (1.53)$$

$$\langle \underset{\sim}{n} \times \underset{\sim}{E} \rangle = 0 \qquad (1.54)$$

where $\underset{\sim}{n}$ is a unit vector normal to the boundary, and the brackets $\langle\,\rangle$ denote the *change* in the bracketted quantity as the boundary is crossed. An alternative form of the boundary conditions is

$$\langle \Phi \rangle = 0 \tag{1.55}$$

$$\langle \underset{\sim}{n} \cdot \underset{\sim}{B} \rangle = 0 \tag{1.56}$$

$$\langle \underset{\sim}{n} \times \underset{\sim}{B} \rangle = 0 \tag{1.57}$$

where Φ is the *electromagnetic scalar potential*. If $\underset{\sim}{A}$ is the *electromagnetic vector potential*, then

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\Phi - \partial\underset{\sim}{A}/\partial t \tag{1.58}$$

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A} \tag{1.58'}$$

When the fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$ vary with time, (1.53) and (1.54) form a complete set of boundary conditions. However, if the fields are time-independent, (1.54) reduces to the single scalar condition (1.55), and the condition (1.56) must be added to give the required number of conditions. Several other conditions can be derived from (1.53)-(1.54) or from (1.55)-(1.57). For example,

$$\langle \underset{\sim}{n} \cdot \underset{\sim}{j} \rangle = 0 \tag{1.59}$$

$$\langle \underset{\sim}{n} \times \underset{\sim}{A} \rangle = 0 \tag{1.59'}$$

(always bearing in mind that $\eta \neq 0$). In general, the magnetic flux density, the tangential electric field, the normal component of electric current density, the scalar potential, and the tangential vector potential are all continuous across a boundary.

An additional condition exists which defines the surface charge density χ at the boundary between two

media "1" and "2".

$$\chi = \langle \epsilon \underset{\sim}{n} \cdot \underset{\sim}{E} \rangle_2^1$$

where the bracket $\langle \, \rangle_2^1$ denotes the amount by which the boundary value of the bracketted quantity in medium 1 exceeds the boundary value of the quantity in medium 2 .

In most cases of interest, we shall assume that the conducting medium is surrounded by a nonconductor which extends to infinity in all directions. The conditions that there be *no sources at infinity* can then be written

$$\begin{aligned} |\underset{\sim}{B}| &= O(r^{-3}) \quad \text{as} \quad r \to \infty \\ |\underset{\sim}{E}| &= O(r^{-2}) \quad \text{as} \quad r \to \infty \qquad (1.60) \\ |\Phi| &= O(r^{-1}) \quad \text{as} \quad r \to \infty \end{aligned}$$

where r is the distance from an origin inside the conducting region.

1.7.2 Mechanical boundary conditions

Six independent scalar "mechanical" boundary conditions must be satisfied. These conditions may be written

$$\langle \underset{\sim}{n} \cdot \underset{\sim}{u} \rangle = 0 \qquad (1.61)$$

$$\langle \underset{\sim}{n} \times \underset{\sim}{u} \rangle = 0 \qquad (1.62)$$

$$\langle n_j p_{ij} \rangle = 0 \qquad (1.63)$$

for a surface of discontinuity separating either two immiscible fluids or a fluid and a solid. For the latter case, if the solid is stationary (1.61) and (1.62) combine

to give the condition

$$\underset{\sim}{u} = 0 \qquad (1.64)$$

on the velocity of the fluid at the interface.

It should be noted that the *no slip* condition (1.62) is an idealization. However, it is widely used because of its simplicity. Strictly speaking, we may only require that the *normal component* of the fluid velocity vanish at a fluid-solid interface. In this thesis we shall follow standard practice and apply the more stringent condition (1.64).

When the solid boundary is rotating, (1.64) is only satisfied *in the rotating frame of reference* - i.e.

$$\underset{\sim}{u}_{rot} = 0 \qquad (1.64')$$

at the fluid-solid interface. Following the practice introduced in section 1.5.3, we shall drop the subscript "rot" and use (1.64') in the form (1.64).

The relation (1.63) provides no useful information when the boundary considered is a fluid-solid interface. The stresses applied by the fluid merely produce elastic strains in the solid.

1.7.3 Thermal boundary conditions

Two independent scalar "thermal" boundary conditions arise. These conditions may be written

$$\langle \lambda \, \partial T/\partial n \rangle = 0 \tag{1.65}$$

$$\langle T \rangle = 0 \tag{1.66}$$

where $\partial T/\partial n$ represents the normal component of the temperature gradient at the boundary. (1.65) and (1.66) imply that both the normal component of the heat conduction vector and the temperature T are continuous across the boundary.

1.8 The full hydromagnetic dynamo problem - body forces

The full hydromagnetic dynamo problem, as it applies to planetary dynamos like that of the Earth, is defined by equations (1.45)-(1.48), (1.51), and (1.52), and by the boundary conditions (1.53)-(1.54), (1.60), and (1.64)-(1.66). The information required before a solution is possible (even in principle) is a complete description of the initial conditions, the heat sources, and the form of the body force density $\underset{\sim}{F}$. Unfortunately, very little of this information is available, and models must be constructed in any attempt to match the observed behaviour of the system. In addition, the full problem is of such formidable difficulty that simplification of the equations is virtually a necessity. We shall discuss some possible simplifications in sections 1.8.4 and 6.1.

One of the first major problems to be considered is the specification of the body force density. Several possibilities have been considered in the literature. We shall discuss these possibilities in the remainder of this section.

1.8.1 Buoyancy forces

The body force density $\underset{\sim}{F}$ may well depend largely on variations of the density of the fluid medium. In the Boussinesq approximation *(Chandrasekhar, 1961, p. 16ff.;*

Jeffreys, 1930; Speigel and Veronis, 1960; P.H. Roberts, 1967a, pp. 194-200) the force density associated with these variations is assumed to be of the *Archimedean* form

$$\underset{\sim}{F}_A = \Theta^* \underset{\sim}{g} \tag{1.67}$$

where $\underset{\sim}{g}$ is the acceleration due to gravity, and Θ^* is proportional to the excess density.

The case in which the density variations are of thermal origin has been studied by several workers *(Chandrasekhar, 1961, Chapters IV and V; Taylor, 1963; Malkus, 1963; Tough and Roberts, 1967; Eltayeb and Roberts, 1970; Soward, 1971a, 1972a,b; Roberts and Soward, 1972; Childress, 1972; Busse, 1972b; G.O. Roberts, 1972b).*

Braginskii (1964d, 1967a) has estimated the contributions to density variations in the Earth's core from thermal and non-thermal sources, and has concluded that Θ^* is due mainly to gravitational separation of lighter elements in the fluid core. He has studied the convection of a two-component fluid, using a generalized form of the dynamo equations. A different model of the same type was suggested by *Urey (1952). See also Artyushkov (1972).*

1.8.2 <u>Precessional torques</u>

If the rotating body of fluid is constrained to precess uniformly with angular velocity $\underset{\sim}{\Omega}'$ about its axis of rotation, an extra term appears on the right hand side

of the Navier-Stokes equation (1.47 or 1.47'). In the frame rotating with angular velocity $\underset{\sim}{\Omega}$, this term has the form

$$\frac{1}{\rho}\underset{\sim}{F}_P = (\underset{\sim}{\Omega}'\times\underset{\sim}{\Omega})\times\underset{\sim}{r} - \frac{1}{2}\underset{\sim}{\nabla}\{[(2\underset{\sim}{\Omega}+\underset{\sim}{\Omega}')\times\underset{\sim}{r}]\cdot[\underset{\sim}{\Omega}'\times\underset{\sim}{r}]\} + (\underset{\sim}{\Omega}'\times\underset{\sim}{r})\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u} + \underset{\sim}{\Omega}'\times\underset{\sim}{u} \qquad (1.68)$$

Malkus (1963, 1968, 1971a,b) has considered the case in which $\underset{\sim}{F}_P$ is the dominant contribution to the body force density. He uses the approximation

$$\frac{1}{\rho}\underset{\sim}{F}_P \approx (\underset{\sim}{\Omega}'\times\underset{\sim}{\Omega})\times\underset{\sim}{r} - \frac{1}{2}\underset{\sim}{\nabla}\{[(2\underset{\sim}{\Omega}+\underset{\sim}{\Omega}')\times\underset{\sim}{r}]\cdot[\underset{\sim}{\Omega}'\times\underset{\sim}{r}]\} \qquad (1.68')$$

The hydrodynamics of viscous flow in a precessing spheroidal cavity have been studied by a number of authors *(Bondi and Lyttleton, 1953; Stewartson and Roberts, 1963; Roberts and Stewartson, 1964). Busse (1968, 1971)* has extended the analysis to the case of a precessing spheroidal shell. The latter geometry is a more appropriate model of the Earth's fluid core because of the presence of the solid inner core. *(See also the review by M.G. Rochester, 1973.)*

1.8.3 Turbulent forces

The *mean field* approach can be applied to the Navier-Stokes equation (1.47, 1.47') as well as to the induction equation (1.16) *(P.H. Roberts, 1971a; Moffatt, 1972)*. In this approach the body force density $\underset{\sim}{F}$ is separated into a

statistical average and a fluctuating part

$$\underset{\sim}{F} = \overline{\underset{\sim}{F}} + \underset{\sim}{F}'$$

The term $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ in the "modified" Ohm's Law (1.17) is then evaluated in terms of the mean fields $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$, and the statistical properties of the fluctuating body force density $\underset{\sim}{F}'$.

Moffatt (1972) has considered a model in which $\underset{\sim}{F}'$ is a homogeneous turbulent field without intrinsic helicity - i.e.

$$\overline{\underset{\sim}{F}' \cdot \mathrm{curl}\, \underset{\sim}{F}'} \equiv 0$$

However, in order to ensure that the fluctuating velocity field $\underset{\sim}{u}'$ does have helicity, he assumes that "...there is some selective mechanism present which leads to a net flux of energy parallel to $\underset{\sim}{\Omega}$ [the angular velocity vector]".

Steenbeck, Krause, and Rädler (1966) have considered a quasi-kinematic dynamo model in which rotation interacts with gradients of density and "turbulent intensity" to produce an α-effect. However, it has been pointed out that the turbulence *spectrum function* used by these authors is not physically realizable *(Lerche, 1972e)*.

The dynamo model of *Steenbeck, Krause, and Rädler (1966)* is not fully *hydromagnetic*, since they consider only gradients of the intensity of the turbulent *velocity* field $\underset{\sim}{u}'$. In Chapter 6 we shall consider the effects due to gradients of the intensity of the turbulent *force density* field $\underset{\sim}{F}'$.

1.8.4 More general studies of the hydromagnetic dynamo problem

In addition to the work mentioned above, there have been several more general studies of the hydromagnetic dynamo problem. *Childress (1968, 1969)* has pointed out that any kinematic dynamo can be used as the basis of a self-consistent hydromagnetic dynamo in any dynamical model simply by choosing the body force density in such a way that the momentum balance equation (1.47) is satisfied. However, in more realistic models where the body force density is specified from the start, the choice of a dynamical model will introduce *consistency conditions* which must be satisfied by $\underset{\sim}{F}$ and $\underset{\sim}{B}$.

Taylor (1963) has derived a particularly interesting consistency condition which has been used by *Thirlby (1972)* as the basis of a numerical study of hydromagnetic dynamo action in a sphere. The Taylor condition arises from the *magnetogeostrophic approximation* to the Navier-Stokes equation (1.47'a) for incompressible flow

$$2\rho\underset{\sim}{\Omega} \times \underset{\sim}{u} = -\underset{\sim}{\nabla}p + \underset{\sim}{j} \times \underset{\sim}{B} + \underset{\sim}{F}^{\dagger} \qquad (1.69)$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0 \qquad (1.70)$$

where p is the dynamic pressure and $\underset{\sim}{F}^{\dagger}$ is a *modified* body force density incorporating the terms $\underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{r})$ and $(\partial\underset{\sim}{\Omega}/\partial t)\times\underset{\sim}{r}$. Equation (1.69) has been obtained from (1.47'a) by neglecting the inertial term $D\underset{\sim}{u}/Dt$ and the viscous term

$\nu\nabla^2\underset{\sim}{u}$ in comparison with the *Coriolis* term $2\underset{\sim}{\Omega} \times \underset{\sim}{r}$. This approximation is valid when the *Rossby* and *Ekman numbers* for the flow are small compared with unity - i.e.

$$Ro \equiv \mathcal{U}/2\Omega L \ll 1 \tag{1.71}$$

$$\varepsilon \equiv \nu/2\Omega L^2 \ll 1 \tag{1.72}$$

- and the time scale of velocity variation is much greater than the rotation period - i.e.

$$1/2\Omega T \ll 1 \tag{1.73}$$

Since equation (1.69) is of lower order than equation (1.47'a), viscous boundary layer theory must be applied to satisfy the boundary conditions on $\underset{\sim}{u}$ *(see section 6.2)*. The boundary-layer thickness δ is of order $\varepsilon^{1/2} L$. The incompressibility condition (1.70) implies that the solution $\underset{\sim}{u}^i$ to (1.69) must satisfy

$$\underset{\sim}{n}\cdot\underset{\sim}{u}^i \sim \varepsilon^{1/2}\, U \tag{1.74}$$

on the boundary S of the conducting fluid, where U is the average magnitude of $\underset{\sim}{u}^i$ on S , and $\underset{\sim}{n}$ is the unit vector normal to S . If $\varepsilon \ll 1$, it is reasonable to assume that

$$\underset{\sim}{n}\cdot\underset{\sim}{u}^i \sim 0 \quad \text{on} \quad S \tag{1.75}$$

Applying Gauss' Theorem to a cylindrical volume V_C coaxial with $\underset{\sim}{\Omega}$ and bounded at the ends by sections of S , we have, from (1.70) and (1.75), that

$$\int_C \underset{\sim}{u}^i \cdot d\underset{\sim}{S} = \int_{V_C} \underset{\sim}{\nabla} \cdot \underset{\sim}{u}^i \, dv - \int_{\substack{\text{portions}\\ \text{of } S}} (\underset{\sim}{n} \cdot \underset{\sim}{u}^i) \, dS$$

$$= 0 \qquad (1.76)$$

where C is the cylindrical side wall of the volume V_C. From (1.76) it follows immediately that

$$2\Omega \int_C \underset{\sim}{u}^i \cdot d\underset{\sim}{S} = \int_C (2\underset{\sim}{\Omega} \times \underset{\sim}{u}^i)_\phi \, dS = 0 \qquad (1.77)$$

where $()_\phi$ represents the azimuthal component. Substituting (1.69) into (1.77) and factoring out the density ρ, we have

$$\int_C (\underset{\sim}{j} \times \underset{\sim}{B})_\phi \, dS = \int_C (\underset{\sim}{\nabla} p - \underset{\sim}{F}^{+})_\phi \, dS \qquad (1.78)$$

If

$$(\underset{\sim}{\nabla} p - \underset{\sim}{F}^{+})_\phi = 0 \quad \text{on } C \qquad (1.79)$$

it follows from (1.78) that

$$\int_C (\underset{\sim}{j} \times \underset{\sim}{B})_\phi \, dS = 0 \qquad (1.80)$$

Equation (1.80) is *Taylor's consistency condition*. It depends on the assumption that (1.70)-(1.73) and (1.79) are all valid. For the geodynamo, (1.71) and (1.72) are certainly true *(see section 6.1)*, and we may certainly choose meaningful time scales for which (1.73) is satisfied. The stratification parameter

$$\Delta\rho/\rho < 0.3 \qquad (1.81)$$

where $\Delta\rho$ represents the change in density over the length

scale L . On the boundary-layer length scale δ , (1.70) is certainly valid, so that (1.75) is a reasonable approximation. Furthermore, the main flow $\underset{\sim}{u}^i$ is predominantly azimuthal, while the density gradient is predominantly radial. It follows that

$$\int_{V_c} \underset{\sim}{\nabla} \cdot \underset{\sim}{u}^i \, dv = - \int_{V_c} \underset{\sim}{u}^i \cdot \frac{\underset{\sim}{\nabla} \rho}{\rho} \, dv \approx 0 \tag{1.82}$$

even though the stratification parameter (1.81) is not much less than unity. The validity of the Taylor condition (1.80) in the geodynamo thus depends mainly on the assumption (1.79). The probability of significant azimuthal body forces in the geodynamo is discussed in *Chapter 6*. *(See also P.H. Roberts, 1971a.)*

Yet another interesting approach to the hydromagnetic dynamo problem has been suggested by *Busse (1973a)*. He attacks the problem by treating the *Lorentz force* term $\frac{1}{\rho}(\underset{\sim}{j} \times \underset{\sim}{B})$ as a perturbation term in the momentum balance equation. In order for this approach to be valid in the case of the geodynamo, it is necessary for the toroidal field in the fluid core to be much smaller than the values of several hundred gauss ($1\ G = 10^{-4}\ T$) commonly assumed. *Busse (1973b)* claims that the toroidal field in the core should be less than an order of magnitude greater than the poloidal field.

1.8.5 Driving forces in the geodynamo

At the present time, there is no general agreement on the type of body force which is appropriate to the geodynamo *(see, for example, Jacobs, 1972b; Malkus, 1972b)*. Objections to thermal convection in the outer core have been raised by *Higgins and Kennedy (1971)* and *Kennedy and Higgins (1973)*, who suggest that the temperature lies close to the melting point of the core material, and that in most of the outer core the melting-point gradient is much less steep than the adiabatic gradient. A temperature distribution of the sort proposed by *Kennedy and Higgins (1973)* would imply that the outer core is stably stratified, except within 200-300 km. of the outer core-inner core boundary. Thermal convection would therefore be restricted to *3-4%* of the volume of the fluid core.

Several arguments have been advanced to counter the proposals of Higgins and Kennedy.

a. Melting curve of pure iron. *Higgins and Kennedy's* estimate of the melting curve for iron may be incorrect, since it is based solely on consideration of the solid phase. *Leppaluoto (1972; see also Verhoogen, 1972)*, using *significant structure theory* of the liquid phase, has obtained a revised estimate of the melting curve of iron under core conditions, and finds that both the melting temperature and the melting-point gradient are

greater than those suggested by *Higgins and Kennedy (1971)*. On the other hand, *Kennedy and Higgins (1973)* argue that significant structure theory may give anomalous results when applied to melting phenomena. *Birch (1972)* has reviewed present knowledge of the melting relations of iron at high pressures, and suggests that "...it appears to be unrealistic to claim that the melting temperature of iron, at core pressures, is known to within 500°".

b. Adiabatic gradient. *Higgins and Kennedy's* estimate of the adiabatic gradient in the outer core may also be incorrect, since it too is based solely on consideration of the solid phase *(Jacobs, 1971a,b, 1972a; Birch, 1972; Kennedy and Higgins, 1973; Frazer, 1973)*. *Jacobs (1971a,b, 1972a)* and *Birch (1972)* have suggested that the adiabatic and melting-point curves in the outer core may be nearly coincident. A relationship of this sort could lead to a state of marginal stability in the outer core *(Jacobs, 1971a,b, 1972a)*.

c. Composition. The arguments of *Higgins and Kennedy (1971)* are based on the properties of pure iron. The presence of a lighter alloying component in the outer core, required by density considerations, may lead to substantial modifications of both the melting-point and the adiabatic gradients *(Jacobs, 1971a,b, 1972a;*

Anderson, 1972a,b; Hall and Murthy, 1972; Birch, 1972; Frazer, 1973; Stewart, 1973). Kennedy and Higgins (1973) argue, however, that the meagre evidence available on the behaviour of saturation curves at high pressures suggests that "...if the adiabatic curve falls on the wrong side of the melting curve of iron, it will be even more on the wrong side of a saturation curve".

d. <u>Radioactive potassium.</u> If the lighter alloying component in the outer core is mainly sulphur *(Murthy and Hall, 1970)*, much of the radionuclide K^{40} present in the Earth will be located in the outer core. This heat source would provide an estimated 10^{12} watts/sec , more than enough to drive thermal convection *(Murthy and Hall, 1972; Goettel, 1972; Jacobs, 1972b; Stacey, 1972)*.

e. <u>"Slurry".</u> Even if the temperature distribution in the outer core does lie along a melting curve with a shallow gradient, it is possible that the core fluid is a *slurry* of fine iron particles suspended in an iron-rich liquid. Under certain conditions, a suspension of this type can behave as an adiabatic fluid *(Busse, 1972; Elsasser, 1972; Malkus, 1972a)*. It is likely, however, that the slurry particles would be unstable *(Malkus, 1972a; Kennedy and Higgins, 1973)*. In addition, seismic evidence on attenuation in the outer core seems to cast doubt on the possibility of a slurry *(Birch, 1972)*.

f. Internal wave motions. Even if the outer core is stably stratified against convection, the geodynamo might still be driven by internal wave motions *(Bullard and Gubbins, 1971, 1973)*.

A further objection to thermally driven models of the geodynamo arises from consideration of the constraints imposed on heat flux in the core by surface heat-flux measurements. There is some question as to whether the thermal energy available will be sufficient to maintain the magnetic field against ohmic loss. There is also some question *(Malkus, 1972b)* as to whether the gravitational separation mechanisms proposed by *Braginskii (1964d, 1967a)* and *Urey (1952)* can supply enough energy to drive the geodynamo.

Malkus (1968, 1971a,b, 1972b) and *Stacey (1973)* claim that there may be just barely enough energy available from precession to drive the geodynamo. However there are substantial objections to several of the arguments used by *Malkus* in support of the precession-driven geodynamo *(Jacobs, Chan, and Frazer, 1972; Jacobs, 1972b; Rochester, et al. 1973)*.

It is difficult to comment on the validity of turbulent force models of the geodynamo without some knowledge of the mechanism by which the turbulent force is assumed to arise. However, it seems unlikely that

turbulence in the core will be homogeneous. For this reason, the model proposed by *Moffatt (1972)* seems inappropriate. Furthermore, Moffatt's requirement of a net energy flow parallel to the rotation axis appears unlikely to be met in the geodynamo. The inhomogeneous model proposed in *section 6.5* of this thesis may provide a more useful approach.

In conclusion, attention should be drawn to several other forces which may well influence the behaviour of the geomagnetic field. It has been suggested *(e.g. Jacobs, 1970b)* that a correlation exists between the frequency of field reversals and the time rate of change of the speed of rotation of the Earth. This suggestion implies that the term $(\partial\underset{\sim}{\Omega}/\partial t) \times \underset{\sim}{r}$ in the Navier-Stokes equation (1.47) may be important. Other forces which may be of interest are those due to oscillations of the Earth's inner core *(Won and Kuo, 1973)*, and those due to roughness of the core-mantle interface *(Hide, 1967; Hide and Horai, 1968; Hide, 1969a; Hide and Malin, 1970, 1971a,b,c; Jacobs, 1971b; Ibrahim, 1973)*.

1.9 The mathematical nature of the dynamo problem

1.9.1 The kinematic dynamo problem

Mathematically speaking, the dynamo problem is extremely complicated. However, the problem has several general features which should be noted. Let us first consider the kinematic dynamo problem. Let ε represent all space, V a simply-connected volume embedded in ε, and S the surface of V. Assume that $\eta = \infty$ in $\varepsilon - V$ and $\eta \neq 0$ in V. The kinematic dynamo problem is then specified by the equations

$$[\partial/\partial t - \eta\nabla^2]\,\underset{\sim}{B} = \underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B}) \quad \text{in} \quad V \qquad (1.83)$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = 0 \quad \text{in} \quad \varepsilon - V \qquad (1.84)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \quad \text{in} \quad \varepsilon \qquad (1.85)$$

and the boundary and initial conditons

$$r^3 B \quad \text{bounded in} \quad \varepsilon \qquad (1.86)$$

$$\langle \underset{\sim}{B} \rangle = 0 \quad \text{on} \quad S \qquad (1.87)$$

$$\underset{\sim}{B}(r,0) = \underset{\sim}{B}_o(r) \qquad (1.88)$$

where $\underset{\sim}{u}$ is an *allowable flow (see section 1.4.1)*. It can be shown *(Backus, 1958; Childress, 1968)* that a solution to the problem (1.83)-(1.88), together with appropriate initial and boundary conditions on $\underset{\sim}{E}$, uniquely determines the solutions to (1.2)-(1.6) and (1.15), provided that the total initial charge on V is known.

The problem can be reduced to one on V alone by noting that $\underset{\sim}{B}$ can be represented by a scalar potential Ψ

in $\varepsilon - V$, where

$$\nabla^2 \Psi = 0 \quad \text{in} \quad \varepsilon - V \tag{1.89}$$

$$r^2 \Psi \quad \text{bounded in} \quad \varepsilon - V \tag{1.90}$$

If the normal component of the gradient of Ψ is specified on S , the external problem (1.89) has a unique solution for Ψ . $\underset{\sim}{B}$ is thus uniquely determined in $\varepsilon - V + S$ by its normal component on S . Because of the continuity condition (1.87) on S , the field in V must satisfy the boundary condition

$$\underset{\sim}{n} \times \underset{\sim}{B} = f_S(\underset{\sim}{n} \cdot \underset{\sim}{B}) \equiv \underset{\sim}{n} \times \underset{\sim}{\nabla} \Psi \quad \text{on} \quad S \tag{1.91}$$

The problem in V is then a linear differential system in $\underset{\sim}{B}$, elliptic in space and parabolic in time. It should be noted that equation (1.85) can be considered as an initial condition *(see the discussion in section 1.3.3)*.

A dimensionless velocity $\underset{\sim}{u}/U$ can be termed a *kinematic dynamo (Childress, 1968)* if it satisfies the conditions:

a) The kinetic energy of the system remains below a specified value

$$E_{kin} \equiv \int_V \tfrac{1}{2} \rho u^2 \, dv \leq (\text{const.}) \cdot \rho U^2 \ , \quad t \geq 0 \tag{1.92}$$

b) There exists a magnetic Reynolds number $R_m^* = UL/\eta$ less than ∞, such that the kinematic dynamo problem has at least one solution $(\underset{\sim}{B}, \underset{\sim}{E})$ for which the magnetic energy M in ε approaches a positive upper limit as $t \to \infty$.

$$\limsup_{t \to \infty} \left\{ M \equiv \int_{\mathcal{E}} \frac{B^2}{2\mu} \, dv \right\} > 0 \tag{1.93}$$

(This condition is somewhat more stringent that the condition suggested in section 1.4.1.)

If $\underset{\sim}{u}$, $\underset{\sim}{B}$, and $\underset{\sim}{E}$ are all independent of time, the kinematic dynamo problem reduces to an elliptic system, linear in $\underset{\sim}{B}$. If the equations are written in nondimensional form, the magnetic Reynolds number appears as an eigenvalue, and the dimensionless velocity $\underset{\sim}{u}/U$ is a *stationary dynamo* if and only if the kinematic dynamo problem has an eigenvalue $R_m^* \neq \infty$ *(Childress, 1968)*.

In the general kinematic problem, there will be a *spectrum* of magnetic-energy growth rates for a given velocity $\underset{\sim}{u}$. The largest possible growth rate is given by the difference between the maximum rate at which energy can be supplied by the interaction between the velocity and magnetic fields, and the minimum rate at which energy can be lost by ohmic dissipation. It can be shown *(P.H. Roberts, 1967b)* that

$$\begin{aligned} \partial M/\partial t &\equiv \frac{\partial}{\partial t} \int_{\mathcal{E}} \frac{B^2}{2\mu} \, dv \\ &= \frac{1}{\mu} \int_V B_i \, e_{ij} \, B_j \, dv - \int_V j^2/\sigma \, dv \end{aligned} \tag{1.94}$$

where

$$e_{ij} \equiv \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\} \tag{1.95}$$

is the rate of strain tensor. No energy is lost by radiation in the quasi-steady approximation.

If $\lambda(\underset{\sim}{r})$ is the largest real eigenvalue of the symmetric matrix e_{ij} at the point $\underset{\sim}{r}$, and Λ is the largest value of $\lambda(\underset{\sim}{r})$ for points in V+S , then

$$\int_V B_i e_{ij} B_j \, dv \le \int_V \lambda(\underset{\sim}{r}) B_i B_i \, dv$$

$$\le \Lambda \int_V B^2 \, dv \le \Lambda \int_\varepsilon B^2 dv$$

$$= 2\mu \Lambda M \qquad (1.96)$$

Similarly, it may be shown *(P.H. Roberts, 1967b)* that

$$\int_V j^2/\sigma \, dv \ge 2CM \qquad (1.97)$$

where C is the smallest possible decay rate for normal modes of the induction equation in a *stationary* volume V . Dimensional arguments show that C is of the form

$$C = (\eta/L^2)\cdot\kappa \qquad (1.98)$$

where $\kappa = \kappa(V)$. *(For a sphere of radius L , $\kappa = \pi^2$.)* Substituting (1.96)-(1.98) into (1.94) we see that

$$\partial M/\partial t \le 2(\Lambda - C)\cdot M \qquad (1.99)$$

or

$$M(t) \le M(0)\, e^{2(\Lambda - C)t} \qquad (1.100)$$

For dynamo action to occur it is necessary that $\Lambda \ge C$ - i.e.

$$\Lambda L^2/\eta \ge \kappa \qquad (1.101)$$

$\Lambda L^2/\eta$ may be interpreted as a magnetic Reynolds number, since Λ has the dimensions U/L .

It must be realized that the elliptic equation (1.83) is defined for an *ensemble* of dimensionless velocities $\underset{\sim}{u}/U$, not all of which are dynamos. It is necessary to decide which of these velocities will belong to the class of dynamos, a problem which involves the generally *nonlinear* correspondence between the existence of an elliptic equation and the nature of its coefficients. This problem is discussed in some detail by *Childress (1968, 1970)*.

Childress (1968) has pointed out several symmetry properties of the kinematic dynamo problem:

a) If $\underset{\sim}{B}(\underset{\sim}{r},t)$ is the solution to the kinematic dynamo problem satisfying the initial condition $\underset{\sim}{B}(\underset{\sim}{r},0) = \underset{\sim}{B}_o(\underset{\sim}{r})$ then $-\underset{\sim}{B}(\underset{\sim}{r},t)$ is the solution satisfying the initial condition $\underset{\sim}{B}(\underset{\sim}{r},0) = -\underset{\sim}{B}_o(\underset{\sim}{r})$.

b) If the volume V is invariant under spatial inversion $(\underset{\sim}{r} \rightarrow -\underset{\sim}{r})$, then the kinematic dynamo problem is invariant under the transformation

$$\begin{aligned} &\{ \underset{\sim}{B}(\underset{\sim}{r},t) \ , \ \underset{\sim}{E}(\underset{\sim}{r},t) \ , \ \underset{\sim}{u}(\underset{\sim}{r},t) \ , \ R_m \} \\ &\quad \rightarrow \{ \underset{\sim}{B}(-\underset{\sim}{r},t) \ , \ \underset{\sim}{E}(-\underset{\sim}{r},t) \ , \ -\underset{\sim}{u}(-\underset{\sim}{r},t) \ , \ R_m \} \end{aligned} \qquad (1.102)$$

c) If (ϖ,ϕ,z) is a system of cylindrical coordinates, the kinematic dynamo problem admits a formal class of solutions satisfying the parity requirements:

$$[B_{\tilde{\omega}}, B_{\phi}, E_z, u_z] \quad \text{odd in } z$$
$$[B_z, E_{\tilde{\omega}}, E_{\phi}, u_{\tilde{\omega}}, u_{\phi}] \quad \text{even in } z \qquad (1.103)$$

The kinematic dynamo equation (1.83) is not *self-adjoint*. However, *Namikawa and Matsushita (1970)* have studied the equation in detail and have shown that it becomes self-adjoint for a curl-free velocity under a suitable restriction. *Lerche (1972c)* has investigated the equations from the *Lagrangian* point of view, and has derived a variational principle for computing the eigenvalues of the dynamo equations subject to the appropriate boundary conditions. He suggests that this method has decided advantages over other numerical techniques which have been used in investigations of the kinematic dynamo problem.

1.9.2 The hydromagnetic dynamo problem

For the hydromagnetic dynamo problem (1.45)-(1.48), the difficulties encountered in the kinematic problem are compounded. We must now solve for $\underset{\sim}{u}$, $\underset{\sim}{B}$, and p , assuming that the body force density $\underset{\sim}{F}$ and the fluid density ρ are given. The implicit nonlinearity introduced by the necessity of determining which of the possible fields $\underset{\sim}{F}$ and ρ lead to dynamo action is severe unless all terms involving $\nabla\rho$ are ignored. In addition, the equations are *explicitly* nonlinear because of the terms $\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}$ and

$\frac{1}{\rho\mu}(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B}$ in the Navier-Stokes equation, and the term $\underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B})$ in the induction equation. Some simplification is obtained if the inertial terms in the Navier Stokes equation are neglected; however, the nonlinear Lorentz force cannot usefully be neglected, since it provides the desired back-reaction of the magnetic field on the flow. The only simplification possible here is to treat the Lorentz force as a perturbation term *(Busse, 1973a; see section 1.8.4)*.

Childress (1968, 1969) has proved an anti-dynamo theorem for the hydromagnetic dynamo problem in the *magnetogeostrophic approximation* (1.69). The theorem may be stated as follows: *if* $\underset{\sim}{u}$ *and* $\underset{\sim}{B}$ *are axisymmetric fields, and* $\underset{\sim}{F} \equiv 0$ *, then* $\underset{\sim}{B}$ *must go to zero as* $t \to \infty$ *, even if an* α*-effect is present.*

Childress (1968) has also pointed out certain symmetry properties of the hydromagnetic dynamo problem. If (ϖ, ϕ, z) is a system of cylindrical coordinates, the hydromagnetic dynamo problem admits a formal class of solutions satisfying the parity requirements:

$$\begin{array}{ll} [F_z, B_\varpi, B_\phi, E_z, u_z] & \text{odd in } z \\ [F_\varpi, F_\phi, B_z, E_\varpi, E_\phi, u_\varpi, u_\phi] & \text{even in } z \end{array} \qquad (1.104)$$

1.10 Summary of Chapter 1

This chapter is concerned with dynamo theory and its application to astrophysical magnetic fields. Most of the material presented is taken from the recent literature on the subject. The principal original contributions are to be found in section 1.1, where a detailed summary of present observational knowledge of astrophysical magnetic fields is presented, and an attempt is made to determine the validity of the *"flux-conserving field compression"* hypothesis concerning magnetic star evolution, and *Schuster's hypothesis* concerning magnetic fields in massive rotating bodies. The data available neither confirm nor disprove the "flux-conserving field compression" hypothesis. Schuster's hypothesis, however, gives erroneous predictions for the dipole fields of the Moon and terrestrial planets other than the Earth.

2. MEAN FIELD ELECTRODYNAMICS

2.1 Introduction

In this thesis we shall be concerned mainly with the "mean field" approach to the solution of the kinematic and hydromagnetic dynamo problems. This approach has received considerable attention in recent years, and extensive reviews of the subject have been written by *P.H. Roberts (1971a), Krause and Rädler (1971), Parker (1970a,b,c, 1971a-f)* and *Lerche and Parker (1971, 1972)*. The East German school *(Steenbeck, Krause, Rädler, et al.)* have introduced the term *mean field electrodynamics* (MFE) to refer to the study of electromagnetic fields in conducting fluids, when the fluids are in turbulent motion.

The standard notation of mean field electrodynamics will be used. *(See, for example, P.H. Roberts, 1971a.)* The fields studied are assumed to have random, or "turbulent" components, so that the concept of a statistical ensemble average is appropriate. A given field F (vector or scalar) can be decomposed into an ensemble average (denoted by an overbar) and a random, or fluctuating component (denoted by a prime).

$$F = \overline{F} + F' \qquad (2.1)$$

These components have the following properties:

$$\overline{F'} = 0 , \qquad \overline{\overline{F}} = \overline{F}$$

$$\overline{F + G} = \overline{F} + \overline{G} , \qquad \overline{\overline{F}G} = \overline{F}\,\overline{G} , \qquad \overline{\overline{F}G'} = 0$$

$$\overline{FG} = \overline{F}\,\overline{G} + \overline{F'G'}$$

In general, a field F will be both position and time-dependent. We shall denote the position vector by $\underset{\sim}{x}$ and the time by t, and write

$$F = F(\underset{\sim}{x},t)$$

2.2 Types of turbulence

2.2.1 General considerations

We shall be dealing with quantities which depend on the joint probability distribution of the values of a field at several points in space and time. *Batchelor (1953, p.20)* defines

$$\overline{u_{a_1}(\underset{\sim}{x}_1,t)u_{a_2}(\underset{\sim}{x}_2,t)\ldots u_{a_m}(\underset{\sim}{x}_m,t)}$$

$$\equiv \bar{Q}^{(m)}_{a_1a_2\ldots a_m}(\underset{\sim}{x}_1;\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_{n-1};t)$$

as an *m-order, n-point product mean value*, where $(\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_{n-1})$ is a $3(n-1)$ dimensional vector specifying the configuration formed by those n of the points $(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_m)$ which are distinct. We shall extend this definition, and refer to

$$\overline{u_{a_1}(\underset{\sim}{x}_1,t_1)u_{a_2}(\underset{\sim}{x}_2,t_2)\ldots u_{a_m}(\underset{\sim}{x}_m,t_m)}$$

$$\equiv \bar{Q}^{(m)}_{a_1a_2\ldots a_m}(\underset{\sim}{x}_1,t_1;\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_{n-1};\tau_1,\ldots,\tau_{q-1}) \tag{2.2}$$

as an *m-order, n-point, q-time product mean value*, where $(\tau_1,\ldots,\tau_{q-1})$ is a $(q-1)$ dimensional vector specifying the configuration formed by those q of the times $(t_1,\ldots,t_m)$ which are distinct.

The invariance properties of the scalar quantity

$$\overline{Q}(\underset{\sim}{x}_1, t_1; \underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_{n-1}; \underset{\sim}{l}_1, \underset{\sim}{l}_2, \ldots, \underset{\sim}{l}_m; \tau_1, \ldots, \tau_{q-1})$$

$$\equiv (\underset{\sim}{l}_1)_{a_1} \cdots (\underset{\sim}{l}_m)_{a_m} \overline{Q}^{(m)}_{a_1 \ldots a_m}(\underset{\sim}{x}_1, t_1; \underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_{n-1}; \tau_1, \ldots, \tau_{q-1}) \tag{2.2'}$$

where the $(\underset{\sim}{l}_i)$ are unit vectors and the Einstein summation convention applies, determine the form of $\overline{Q}^{(m)}_{a_1 \ldots a_m}$ *(Robertson, 1940; Batchelor, 1953, pp. 40-45).* The invariance properties of interest are those which refer to spatial and temporal translations, rotations, and reflections of the configuration of points and times defining the average, combined with the unit vectors $(\underset{\sim}{l}_i)$. The various types of turbulence can be characterized by their invariance properties, and special terminology has been introduced to describe the simplest types of invariance.

a. <u>Stationary turbulence.</u> $\overline{Q}$ is invariant with respect to arbitrary temporal translations of the configuration of times $(t_1, \ldots, t_m)$. Therefore both $\overline{Q}$ and $\overline{Q}^{(m)}_{a_1 \ldots a_m}$ are independent of t_1 .

b. <u>Homogeneous turbulence.</u> $\overline{Q}$ is invariant with respect to arbitrary spatial translations of the configuration of points $(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m)$. Therefore both $\overline{Q}$ and $\overline{Q}^{(m)}_{a_1 \ldots a_m}$ are independent of $\underset{\sim}{x}_1$.

Once homogeneity is assumed, further conditions on the spatial properties of the turbulence can be introduced.

In each of the next six definitions, the term *homogeneous* is omitted, although it is to be understood.

c. <u>Isotropic turbulence.</u> $\overline{Q}$ is invariant with respect to arbitrary spatial reflections and rigid-body rotations of the configuration $(\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_{n-1};\underset{\sim}{l}_1,\ldots,\underset{\sim}{l}_m)$. When homogeneity is taken into account, we see that $\overline{Q}$ is invariant under the extended group of roto-translations in 3-dimensional space (this group includes spatial reflections as part of the rotation group).

Definitions (a)-(c) agree with those given by *Batchelor (1953, p. 18 and p. 41)*. Unfortunately, this terminology is not universally accepted. For example, because it is often convenient to separate the concepts of reflection and rotation invariance, some authors do not include reflection symmetry in their definition of *isotropy (e.g. P.H. Roberts, 1971a)*. The term *isotropic, mirror-symmetric turbulence* is then introduced to refer to what *Batchelor* calls *isotropic turbulence (e.g. Krause and Roberts, 1973)*. The term *statistically steady turbulence* provides another example. In some cases *(e.g. P.H. Roberts, 1971a)* this term is used in place of *stationary turbulence*, while in others *(e.g. Krause and Roberts, 1973)*, it is used in place of *stationary, homogeneous turbulence*. In this thesis we shall retain the classical definitions of *stationarity*, *homogeneity*, and *isotropy*. However, we shall

also use terms taken from elementary particle theory *(Fonda and Ghirardi, 1970)* and crystallography *(Shubnikov, 1951)* to describe other types of invariance.

d. "P-invariant" (*or* "mirror-symmetric") turbulence.

$\overline{Q}$ is invariant under spatial inversion, or "reflection in a point" *(Shubnikov, 1951)* of the configuration $(\underset{\sim}{r}_i;\underset{\sim}{l}_j)$. We have used the notation

$$(\underset{\sim}{r}_i;\underset{\sim}{l}_j) \equiv (\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_{n-1};\underset{\sim}{l}_1,\ldots,\underset{\sim}{l}_m)$$

This operation consists of a 180° rotation about some axis, followed by reflection in the plane normal to the axis of rotation. When homogeneity is taken into account, $\overline{Q}$ is invariant under the group of translations and spatial inversions, but not necessarily invariant under the rotation group or any of its subgroups (other than the identity, of course). The term *P-invariance* is taken from elementary particle theory *(Fonda and Ghirardi, 1970, p. 89 and p. 395)*, with "P" standing for *parity*.

e. "R-invariant" (*or* "pseudo-isotropic") turbulence.

$\overline{Q}$ is invariant under rigid-body spatial rotations, but not necessarily invariant under spatial reflections. When homogeneity is taken into account, $\overline{Q}$ is invariant under the *restricted* group of rototranslations (which excludes spatial inversion), but not necessarily invariant under the extended group. Again, the term

R-invariance is taken from elementary particle theory, though with less justification than the term *P-invariance*, since we are not considering rotations in four-dimensional space-time. The "R" stands for *rotation*.

f. "Skew-isotropic" turbulence. $\bar{Q}$ is invariant under rigid-body rotations, but *changes sign* under spatial inversion.

g. "Axially-symmetric" (*or* "axisymmetric") turbulence. $\bar{Q}$ is invariant under spatial rotations of $(\underset{\sim}{r}_i;\underset{\sim}{l}_j)$ about a given unit vector $\underset{\sim}{\lambda}$, and invariant under spatial inversion. When homogeneity is taken into account, $\bar{Q}$ is invariant under the extended group of rototranslations applied to the *augmented* configuration $(\underset{\sim}{\lambda},\underset{\sim}{r}_i;\underset{\sim}{l}_j)$.

h. "Axially R-invariant" turbulence. $\bar{Q}$ is invariant under the restricted group of rototranslations applied to the augmented configuration defined in (g), but not necessarily invariant under the extended group of rototranslations.

In a similar fashion, further conditions on the temporal properties of the turbulence can be introduced once *stationarity* is assumed.

i. "T-invariant" turbulence. $\overline{Q}$ is invariant under the operation of time reversal applied to the times $(t_1, \ldots, t_m)$. Thus $\overline{Q}$ is invariant under the transformation

$$(\tau_1, \ldots, \tau_{q-1}) \rightarrow (-\tau_1, \ldots, -\tau_{q-1})$$

Taking stationarity into account, $\overline{Q}$ is invariant under the group of temporal translations and inversions applied to the configuration (τ_k) , where

$$(\tau_k) \equiv (\tau_1, \ldots, \tau_{q-1})$$

The term *T-invariance* is taken from elementary particle theory *(Fonda and Ghirardi, 1970, pp. 104ff.)*.

The types of invariance defined in (d)-(h) can of course occur in combination with the type defined in (i). There are also types of invariance which refer to spatial and temporal transformations applied *simultaneously* to the configuration $(\underset{\sim}{r}_i; \underset{\sim}{l}_j; \tau_k)$. We shall be particularly concerned with

j. "PT-invariant" turbulence. $\overline{Q}$ is invariant under the combined operations of spatial inversion and time reversal, *but not necessarily under either of them separately*. Again, the term *PT-invariance* is taken from elementary particle theory.

The various types of turbulence defined above are listed, with their properties, in Table 11 *(see p. 90)*.

2.2.2 Two-point, two-time correlations

In this thesis we shall be concerned mainly with two-point, two-time correlations. From (2.2),

$$\overline{Q}^{(2)}_{ij}(\underset{\sim}{x}_1,t_1;\underset{\sim}{r},\tau) = \overline{u_i(\underset{\sim}{x}_1,t_1)\; u_j(\underset{\sim}{x}_1+\underset{\sim}{r},\; t_1+\tau)}$$

We shall use the standard notation

$$\overline{Q}^{(2)}_{ij}(\underset{\sim}{x}_1,t_1;\underset{\sim}{r},\tau) \equiv R_{ij}(\underset{\sim}{x}_1,t_1;\underset{\sim}{r},\tau) \tag{2.3}$$

From the definition of the *two-point, two-time correlation tensor* R_{ij} we see that

$$R_{ij}(\underset{\sim}{x}_1,t_1;\underset{\sim}{r},\tau) = R_{ji}(\underset{\sim}{x}_1+\underset{\sim}{r},\; t_1+\tau;\; -\underset{\sim}{r},\; -\tau) \tag{2.4}$$

In the case of homogeneous, stationary turbulence, the dependence of R_{ij} on $\underset{\sim}{x}_1$ and t_1 drops out, and (2.4) becomes

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ji}(-\underset{\sim}{r},-\tau) \tag{2.4'}$$

We also have that

$$\overline{Q}(\underset{\sim}{r},\underset{\sim}{l}_1,\underset{\sim}{l}_2,\tau) = (\underset{\sim}{l}_1)_i\,(\underset{\sim}{l}_2)_j\; R_{ij}(\underset{\sim}{r},\tau) \tag{2.5}$$

The operation of spatial inversion is represented by the transformation

$$(\underset{\sim}{r},\underset{\sim}{l}_1,\underset{\sim}{l}_2,\tau) \rightarrow (-\underset{\sim}{r},-\underset{\sim}{l}_1,-\underset{\sim}{l}_2,\tau)$$

Clearly,

$$\begin{aligned}\bar{Q}(-\underset{\sim}{r},-\underset{\sim}{l}_1,-\underset{\sim}{l}_2,\tau) &= (-1)^2(\underset{\sim}{l}_1)_i(\underset{\sim}{l}_2)_j\, R_{ij}(-\underset{\sim}{r},\tau) \\ &= \bar{Q}(-\underset{\sim}{r},\underset{\sim}{l}_1,\underset{\sim}{l}_2,\tau)\end{aligned}$$

It follows immediately that the property of *P-invariance* implies that

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ij}(-\underset{\sim}{r},\tau) \qquad [\textit{P-invariance}] \qquad (2.6)$$

Combining this property with (2.4'), we have that

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ji}(\underset{\sim}{r},-\tau) \qquad [\textit{P-invariance}] \qquad (2.6')$$

In a similar fashion, we may show that *T-invariance* implies

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ij}(\underset{\sim}{r},-\tau) \qquad [\textit{T-invariance}] \qquad (2.7)$$

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ji}(-\underset{\sim}{r},\tau) \qquad [\textit{T-invariance}] \qquad (2.7')$$

while *PT-invariance* implies

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ij}(-\underset{\sim}{r},-\tau) \qquad [\textit{PT-invariance}] \qquad (2.8)$$

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ji}(\underset{\sim}{r},\tau) \qquad [\textit{PT-invariance}] \qquad (2.8')$$

It will be noted that while P- and T-invariance combine to give PT-invariance, PT-invariance does not necessarily imply either P-invariance or T-invariance. It will also be noted

that PT-invariance implies *symmetry* of the tensor R_{ij} under interchange of its indices. This property will be considered in greater detail in *Chapter 3*.

TABLE 11 - INVARIANCE PROPERTIES OF DIFFERENT TYPES OF TURBULENCE

Type of Turbulence	Space Translation	Space Rotation	Space Inversion	Time Translation	Time Inversion	Space-Time Inversion
HOMOGENEOUS	*invariant*	-	-	-	-	-
R-invariant	*invariant*	*invariant*	-	-	-	-
P-invariant	*invariant*	-	*invariant*	-	-	-
Isotropic	*invariant*	*invariant*	*invariant*	-	-	-
Skew-isotropic	*invariant*	*invariant*	*changes sign*	-	-	-
Axisymmetric	*invariant*	*invariant about axis*	*invariant*	-	-	-
Axially R-invariant	*invariant*	*invariant about axis*	-	-	-	-
STATIONARY	-	-	-	*invariant*	-	-
T-invariant	-	-	-	*invariant*	*invariant*	-
STATIONARY AND HOMOGENEOUS	*invariant*	-	-	*invariant*	-	-
PT-invariant	*invariant*	-	-	*invariant*	-	*invariant*
P-invariant	*invariant*	-	*invariant*	*invariant*	-	-
T-invariant	*invariant*	-	-	*invariant*	*invariant*	-
R-invariant	*invariant*	*invariant*	-	*invariant*	-	-
Isotropic	*invariant*	*invariant*	*invariant*	*invariant*	*invariant**	*invariant**

* *For two-point, two-time correlations.*

2.3 The kinematic dynamo problem - Green's function techniques

2.3.1 The dynamo equations

We shall now apply the mean field approach to the kinematic dynamo problem. Writing $\underset{\sim}{u}$ and $\underset{\sim}{B}$ in the form (2.1), substituting these expressions into the induction equation (1.16'), and separating the average and fluctuating parts of the equation, we obtain the coupled system of equations

$$\{\partial/\partial t - \eta\nabla^2\}\,\overline{\underset{\sim}{B}} - \mathrm{curl}\{\overline{\underset{\sim}{u}} \times \overline{\underset{\sim}{B}}\} = \mathrm{curl}\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}\} \tag{2.9}$$

$$\{\partial/\partial t - \eta\nabla^2\}\,\underset{\sim}{B}' - \mathrm{curl}\{\overline{\underset{\sim}{u}} \times \underset{\sim}{B}'\}$$
$$= \mathrm{curl}\{\underset{\sim}{u}' \times \overline{\underset{\sim}{B}} + \underset{\sim}{u}' \times \underset{\sim}{B}' - \overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}\} \tag{2.10}$$

where

$$\mathrm{div}\,\overline{\underset{\sim}{B}} = \mathrm{div}\,\underset{\sim}{B}' = 0 \tag{2.11}$$

2.3.2 The first-order solution to the fluctuating induction equation

As it stands, equation (2.10) cannot be solved directly for $\underset{\sim}{B}'$ as a functional of the mean fields $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$. However, if the *second order term* in the fluctuating fields

$$\underset{\sim}{C}' \equiv \underset{\sim}{u}' \times \underset{\sim}{B}' - \overline{\underset{\sim}{u}' \times \underset{\sim}{B}'} \tag{2.12}$$

is neglected to a first approximation, (2.10) reduces to

$$\{\partial/\partial t - \eta\nabla^2\}\,\underset{\sim}{B}'_o \;-\; \mathrm{curl}\,\{\underset{\sim}{\overline{u}} \times \underset{\sim}{B}'_o\} \;=\; \mathrm{curl}\,\{\underset{\sim}{u}' \times \underset{\sim}{\overline{B}}\} \tag{2.13}$$

This equation can be solved by Green's function techniques, the right hand side being regarded as a source term *(Rädler, 1964, 1968a; Krause, 1968a,b; Krause and Rädler, 1971).* The solution is of the form

$$\begin{aligned}\underset{\sim}{B}'_o(\underset{\sim}{x},t) = {} & \int_V d\underset{\sim}{x}'\; \underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t_o)\cdot\underset{\sim}{B}'_o(\underset{\sim}{x}',t_o) \\ & + \int_{t_o}^{t} dt' \int_V d\underset{\sim}{x}'\; \underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\cdot \mathrm{curl}\,\{\underset{\sim}{u}'(\underset{\sim}{x}',t')\times\underset{\sim}{\overline{B}}(\underset{\sim}{x}',t')\} \\ & + \eta\int_{t_o}^{t} dt' \int_S dS' \Big\{ \underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\cdot\frac{\partial}{\partial n'}\underset{\sim}{B}'_o(\underset{\sim}{x}',t') \\ & \qquad - \frac{\partial}{\partial n'}\underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\cdot\underset{\sim}{B}'_o(\underset{\sim}{x}',t')\Big\}\end{aligned} \tag{2.14}$$

where t_o is the initial instant, V the region occupied by the turbulence, and S the boundary surface of V. $\partial/\partial n'$ represents a normal derivative on S. $\underset{\approx}{G}$ is a *Green's dyadic* satisfying

$$\begin{aligned}\{\partial/\partial t - \eta\nabla^2\}\,\underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') - \mathrm{curl}\,\{\underset{\sim}{\overline{u}}(\underset{\sim}{x},t)\times\underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\} \\ = \underset{\approx}{I}\;\delta(\underset{\sim}{x}-\underset{\sim}{x}')\,\delta(t-t')\end{aligned} \tag{2.15}$$

$$\lim_{t\to t'} \underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \underset{\approx}{I}\;\delta(\underset{\sim}{x}-\underset{\sim}{x}') \tag{2.16}$$

$$\underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = 0 \quad \text{if} \quad t<t' \tag{2.17}$$

In (2.15) and (2.16) $\underset{\approx}{I}$ is the *idemfactor*, or *"unity" dyadic (see Morse and Feshbach, 1953, p. 57)*, while $\delta(\underset{\sim}{x}-\underset{\sim}{x}')$ and $\delta(t-t')$ are Dirac delta functions. In deriving (2.14) use has been made of the boundary condition

$$\underset{\sim}{u}'(\underset{\sim}{x},t) = 0 , \quad \underset{\sim}{x} \in S \tag{2.18}$$

Boundary conditions must also be specified for $\underset{\sim}{B}'_o$. For simplicity, we shall assume that

$$\underset{\sim}{B}'_o(\underset{\sim}{x},t) = 0 , \quad \underset{\sim}{x} \in S \tag{2.19}$$

and

$$\underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = 0 , \quad \underset{\sim}{x} \in S \text{ or } \underset{\sim}{x}' \in S \tag{2.20}$$

so that the surface term in (2.14) vanishes identically. Finally, as noted in the introduction, the second part of (2.11) is satisfied for all $t \geq t_o$ if

$$\operatorname{div} \underset{\sim}{B}'_o(\underset{\sim}{x},t_o) = 0 \tag{2.21}$$

From (2.14) we can evaluate the source term $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'_o}$ in (2.9) as a functional of $\overline{\underset{\sim}{B}}$, $\overline{\underset{\sim}{u}}$, the statistical properties of $\underset{\sim}{u}'$, and the initial conditions on $\overline{\underset{\sim}{u}'\underset{\sim}{B}'_o}$. In tensor notation,

$$\begin{aligned}
&\{\overline{\underset{\sim}{u}'(\underset{\sim}{x},t) \times \underset{\sim}{B}'_o(\underset{\sim}{x},t)}\}_i \\
&\quad = \epsilon_{ijk} \int_V d\underset{\sim}{x}'\, G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}',t_o)\, \overline{u'_j(\underset{\sim}{x},t)\, B'_{o\ell}(\underset{\sim}{x}',t_o)} \\
&\quad + \epsilon_{ijk}\,\epsilon_{\ell mn}\,\epsilon_{npq} \int_V d\underset{\sim}{x}' \int_{t_o}^{t} dt'\, G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}',t') \\
&\qquad\qquad \cdot \frac{\partial}{\partial x'_m}\{R_{jp}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\,\overline{B}_q(\underset{\sim}{x}',t')\}
\end{aligned} \tag{2.22}$$

where R_{ij} is the two-point, two-time correlation tensor for $\underset{\sim}{u}'$ *(see section 2.2.2)*,

$$R_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \overline{u_i'(\underset{\sim}{x},t)\, u_j'(\underset{\sim}{x}',t')} \tag{2.23}$$

Clearly, if

$$\overline{u_i'(\underset{\sim}{x},t)\, B_{oj}'(\underset{\sim}{x}',t_o)} = 0 \,, \quad \forall\ \underset{\sim}{x},\underset{\sim}{x}' \in V \,,\ t \geqslant t_o \tag{2.24}$$

then $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}$ depends only on $\overline{\underset{\sim}{B}}$, $\overline{\underset{\sim}{u}}$, and the statistical properties of $\underset{\sim}{u}'$.

2.3.3 Higher-order terms

The full solution to (2.10) may now be developed by iteration *(Krause, 1968a,b)*. We write

$$\underset{\sim}{B}' = \underset{\sim}{B}_o' + \underset{\sim}{B}_1' + \underset{\sim}{B}_2' + \ldots \tag{2.25}$$

$$\underset{\sim}{C}_n' = \underset{\sim}{u}' \times \underset{\sim}{B}_n' - \overline{\underset{\sim}{u}' \times \underset{\sim}{B}_n'} \tag{2.26}$$

Substituting (2.25) and (2.26) into (2.10), and using (2.14) as the first term in the solution, we have

$$\left\{ \partial/\partial t - \eta\nabla^2 \right\} \underset{\sim}{B}_n' - \mathrm{curl}\left\{ \overline{\underset{\sim}{u}} \times \underset{\sim}{B}_n' \right\} = \mathrm{curl}\, \underset{\sim}{C}_{n-1}' \,, \quad n \geqslant 1 \tag{2.27}$$

$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ can then be evaluated from

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'} = \overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'} + \overline{\underset{\sim}{u}' \times \underset{\sim}{B}_1'} + \overline{\underset{\sim}{u}' \times \underset{\sim}{B}_2'} + \cdots \tag{2.28}$$

Writing the solution to (2.27) out explicitly,

$$
\begin{aligned}
B'_n(\underset{\sim}{x},t) &= \int_V d\underset{\sim}{x}' \; \underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t_0)\cdot \underset{\sim}{B}'_n(\underset{\sim}{x}',t_0) \\
&\quad + \int_V d\underset{\sim}{x}' \int_{t_0}^{t} dt' \; \underset{\approx}{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\cdot \operatorname{curl} \underset{\sim}{C}'_{n-1}(\underset{\sim}{x}',t')
\end{aligned}
\tag{2.29}
$$

so that

$$
\begin{aligned}
\{\underset{\sim}{u}'\times\underset{\sim}{B}'_n\}_i &= \epsilon_{ijk}\int_V d\underset{\sim}{x}' \; G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}',t_0)\, u'_j(\underset{\sim}{x},t)\, B'_{n\ell}(\underset{\sim}{x}',t_0) \\
&\quad + \epsilon_{ijk}\,\epsilon_{\ell mq}\int_V d\underset{\sim}{x}' \int_{t_0}^{t} dt' \; G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}',t') \\
&\qquad\qquad \cdot \frac{\partial}{\partial x'_m}\left\{ u'_j(\underset{\sim}{x},t)\, C'_{n-1\,q}(\underset{\sim}{x}',t')\right\}
\end{aligned}
\tag{2.30}
$$

Krause (1968a,b) has shown that the iteration process is convergent for all times t and for any choice of $\underset{\sim}{u}$, when V is an infinite domain.

2.4 The kinematic dynamo problem - Fourier transform techniques

2.4.1 Notation

Results similar to those derived in *section 2.3* can be obtained by using *Fourier-Stieltjes transforms* of the fluctuating fields $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ *(Batchelor, 1953, pp. 28-33; Moffatt, 1970a).*

The *Fourier-Stieltjes* representation of a random vector function $\underset{\sim}{f}'(\underset{\sim}{x},t)$ is of the form

$$\underset{\sim}{f}'(\underset{\sim}{x},t) = \int_{\underset{\sim}{k}} \int_{\omega} d\underset{\sim}{F}(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x} + \omega t\}} \qquad (2.31)$$

This type of representation is appropriate when the integral

$$\int d\underset{\sim}{x} \int dt \; |\underset{\sim}{f}'(\underset{\sim}{x},t)|$$

taken over the whole field is not bounded. If $\underset{\sim}{f}'(\underset{\sim}{x},t)$ is a stationary, homogeneous random function - i.e. if the two-point, two-time correlation tensor

$$Q_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \overline{f_i'(\underset{\sim}{x},t)\, f_j'(\underset{\sim}{x}',t')} \qquad (2.32)$$

satisfies the condition

$$Q_{ij}(\underset{\sim}{x},t;\, \underset{\sim}{x}+\underset{\sim}{r},\, t+\tau) = Q_{ij}(\underset{\sim}{r},\tau) \qquad (2.33)$$

- then

$$\overline{dF_i^*(\underset{\sim}{k},\omega)\, dF_j(\underset{\sim}{k}',\omega')}$$

$$= \left\{ \frac{1}{(2\pi)^4} \int d\underset{\sim}{r} \int d\tau \;\; Q_{ij}(\underset{\sim}{r},\tau)\, e^{-i\{\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega\tau\}} \right\}$$

$$\cdot\; \delta(\underset{\sim}{k}-\underset{\sim}{k}')\,\delta(\omega-\omega')\; d\underset{\sim}{k}\, d\underset{\sim}{k}'\, d\omega\, d\omega'$$

$$\equiv \;\; \chi_{ij}(\underset{\sim}{k},\omega)\;\; \delta(\underset{\sim}{k}-\underset{\sim}{k}')\,\delta(\omega-\omega')\; d\underset{\sim}{k}\, d\underset{\sim}{k}'\, d\omega\, d\omega' \qquad (2.34)$$

The Fourier transform

$$\chi_{ij}(\underset{\sim}{k},\omega) \equiv \frac{1}{(2\pi)^4} \int d\underset{\sim}{r} \int d\tau \;\; Q_{ij}(\underset{\sim}{r},\tau)\, e^{-i\{\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega\tau\}} \qquad (2.34')$$

appearing in (2.34) is referred to as the *spectrum tensor* of $\underset{\sim}{f}'(\underset{\sim}{x},t)$.

2.4.2 The fluctuating induction equation

The Fourier-Stieltjes representations of $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ will be written

$$\underset{\sim}{u}'(\underset{\sim}{x},t) \;=\; \int_{\underset{\sim}{k}} \int_{\omega} d\underset{\sim}{Z}(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega\tau\}} \qquad (2.35)$$

$$\underset{\sim}{B}'(\underset{\sim}{x},t) \;=\; \int_{\underset{\sim}{k}} \int_{\omega} d\underset{\sim}{Y}(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega\tau\}} \qquad (2.36)$$

following *Moffatt (1970a)*. The expansion (2.25) can now be replaced by the expansion

$$d\underset{\sim}{Y} \;=\; d\underset{\sim}{Y}_o + d\underset{\sim}{Y}_1 + d\underset{\sim}{Y}_2 + \ldots \qquad (2.37)$$

where

$$\underset{\sim}{B}'_n(\underset{\sim}{x},t) = \int_{\underset{\sim}{k}} \int_{\omega} d\underset{\sim}{Y}_n(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x} + \omega t\}} \tag{2.38}$$

Substituting (2.36) and (2.38) into (2.11),

$$i\underset{\sim}{k}\cdot d\underset{\sim}{Y} = i\underset{\sim}{k}\cdot d\underset{\sim}{Y}_n = 0 \tag{2.39}$$

Before the Fourier transform of (2.10), (2.13), or (2.27) can be obtained, we must define Fourier representations of $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$.

$$\overline{\underset{\sim}{u}}(\underset{\sim}{x},t) = \int_{\underset{\sim}{K}} \int_{\Omega} \hat{\overline{\underset{\sim}{u}}}(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x} + \Omega t\}}\, d\underset{\sim}{K}\, d\Omega \tag{2.40}$$

$$\overline{\underset{\sim}{B}}(\underset{\sim}{x},t) = \int_{\underset{\sim}{K}} \int_{\Omega} \hat{\overline{\underset{\sim}{B}}}(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x} + \Omega t\}}\, d\underset{\sim}{K}\, d\Omega \tag{2.41}$$

Substituting (2.35), (2.38), (2.40), and (2.41) into (2.13) and taking the Fourier transform, we obtain

$$\begin{aligned} &\{i\omega + \eta k^2\}\, d\underset{\sim}{Y}_o(\underset{\sim}{k},\omega) \\ &\qquad - i\underset{\sim}{k} \times \int_{\underset{\sim}{K}} \int_{\Omega} \{\hat{\overline{\underset{\sim}{u}}}(\underset{\sim}{K},\Omega) \times d\underset{\sim}{Y}_o(\underset{\sim}{k}-\underset{\sim}{K},\omega-\Omega)\}\, d\underset{\sim}{K}\, d\Omega \\ &= i\underset{\sim}{k} \times \int_{\underset{\sim}{K}} \int_{\Omega} \{d\underset{\sim}{Z}(\underset{\sim}{k}-\underset{\sim}{K},\omega-\Omega) \times \hat{\overline{\underset{\sim}{B}}}(\underset{\sim}{K},\Omega)\}\, d\underset{\sim}{K}\, d\Omega \end{aligned} \tag{2.42}$$

This equation must be solved for $d\underset{\sim}{Y}_o(\underset{\sim}{k},\omega)$ as a functional of $d\underset{\sim}{Z}(\underset{\sim}{k},\omega)$ and the Fourier transforms of the mean fields, taking into account the initial conditions on $\underset{\sim}{B}'$.

2.4.3 Solution of the fluctuating induction equation for uniform, time-independent mean fields

Equation (2.42) has no simple solution as it stands, because of its complicated dependence on the mean fields. However, if it can be assumed that the mean fields are effectively uniform and time-independent on the length and time scales of the fluctuating fields, we may write

$$\hat{\underset{\sim}{u}}(\underset{\sim}{K},\Omega) \simeq \hat{\underset{\sim}{u}}(0,0)\ \delta(\underset{\sim}{K})\ \delta(\Omega) \tag{2.43}$$

$$\hat{\underset{\sim}{B}}(\underset{\sim}{K},\Omega) \simeq \hat{\underset{\sim}{B}}(0,0)\ \delta(\underset{\sim}{K})\ \delta(\Omega) \tag{2.44}$$

Under this assumption, (2.42) reduces to

$$\begin{aligned}\{i\omega + \eta k^2\}\, d\underset{\sim}{Y}_0(\underset{\sim}{k},\omega) \;-\; i\underset{\sim}{k}\times\{\hat{\underset{\sim}{u}}(0,0)\times d\underset{\sim}{Y}_0(\underset{\sim}{k},\omega)\} \\ = i\underset{\sim}{k}\times\{d\underset{\sim}{Z}(\underset{\sim}{k},\omega)\times\hat{\underset{\sim}{B}}(0,0)\}\end{aligned} \tag{2.45}$$

The solution of this equation is

$$\begin{aligned}d\underset{\sim}{Y}_0(\underset{\sim}{k},\omega) = \frac{i\{\underset{\sim}{k}\cdot\hat{\underset{\sim}{B}}(0,0)\,d\underset{\sim}{Z}(\underset{\sim}{k},\omega) - \underset{\sim}{k}\cdot d\underset{\sim}{Z}(\underset{\sim}{k},\omega)\,\hat{\underset{\sim}{B}}(0,0)\}}{\{i\omega + \eta k^2 + i\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}}(0,0)\}} \\ + d\underset{\sim}{Y}_0'(\underset{\sim}{k})\ \delta(\omega - i\eta k^2)\end{aligned} \tag{2.46}$$

where the term involving $d\underset{\sim}{Y}_o'(k)$ has been added to take the initial conditions into account. We may then write an expression for $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}$:

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'} = \iiiint\limits_{\underset{\sim}{k}\omega\underset{\sim}{k}'\omega'} \overline{d\underset{\sim}{Z}^*(\underset{\sim}{k},\omega) \times d\underset{\sim}{Y}_o(\underset{\sim}{k}',\omega')}\, e^{i\{(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} + (\omega-\omega')t\}}$$

$$= \iiiint\limits_{\underset{\sim}{k}\omega\underset{\sim}{k}'\omega'} \{i\omega + \eta k^2 + i\underset{\sim}{k}\cdot\hat{\underset{\sim}{U}}(0,0)\}^{-1} \cdot i\Big\{\underset{\sim}{k}'\cdot\hat{\underset{\sim}{B}}(0,0)\, \overline{d\underset{\sim}{Z}^*(\underset{\sim}{k},\omega) \times d\underset{\sim}{Z}(\underset{\sim}{k}',\omega')} - \underset{\sim}{k}'\cdot\overline{d\underset{\sim}{Z}(\underset{\sim}{k}',\omega')\, d\underset{\sim}{Z}^*(\underset{\sim}{k},\omega)} \times \hat{\underset{\sim}{B}}(0,0)\Big\} \cdot e^{i\{(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} + (\omega'-\omega)t\}}$$

$$+ \iiint\limits_{\underset{\sim}{k}\omega\underset{\sim}{k}'} \overline{d\underset{\sim}{Z}^*(\underset{\sim}{k},\omega) \times d\underset{\sim}{Y}_o'(\underset{\sim}{k}')}\, e^{i\{(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} + (i\eta k'^2-\omega)t\}} \tag{2.47}$$

Assuming that $\underset{\sim}{u}'$ is a stationary, homogeneous random function, and denoting the *spectrum tensor* of $\underset{\sim}{u}'$ by $\Phi_{ij}(k,\omega)$, we have from (2.34) that

$$\overline{dZ_i^*(\underset{\sim}{k},\omega)\, dZ_j(\underset{\sim}{k}',\omega')}$$

$$= \Phi_{ij}(\underset{\sim}{k},\omega)\,\delta(\underset{\sim}{k}-\underset{\sim}{k}')\,\delta(\omega-\omega')\; d\underset{\sim}{k}\, d\underset{\sim}{k}'\, d\omega\, d\omega' \tag{2.48}$$

Thus

$$\{\overline{\underset{\sim}{u}'\times\underset{\sim}{B}_o'}\}_i = i\epsilon_{ij\ell}\iint\limits_{\underset{\sim}{k}\omega} \frac{\{k_m\Phi_{j\ell}(\underset{\sim}{k},\omega)\,\hat{\hat{B}}_m(0,0) - k_m\Phi_{jm}(\underset{\sim}{k},\omega)\,\hat{\hat{B}}_\ell(0,0)\}}{\{i\omega + \eta k^2 + i\underset{\sim}{k}\cdot\hat{\underset{\sim}{U}}(0,0)\}}\, d\underset{\sim}{k}\, d\omega$$

$$+ \iiint\limits_{\underset{\sim}{k}\omega\underset{\sim}{k}'} \overline{d\underset{\sim}{Z}^*(\underset{\sim}{k},\omega) \times d\underset{\sim}{Y}_o'(\underset{\sim}{k}')}\, e^{i\{(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} + (i\eta k'^2-\omega)t\}} \tag{2.49}$$

2.4.4 Solution of the fluctuating induction equation for nearly uniform, nearly time-independent mean fields

An alternative method of deriving equation (2.49) is to substitute (2.35) and (2.38) directly into (2.13) *without* using a Fourier representation of $\bar{\underset{\sim}{u}}$ and $\bar{\underset{\sim}{B}}$. The equation for $d\underset{\sim}{Y}_o$ then becomes

$$\{i\omega + \eta k^2 + i\,\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}} + \operatorname{div}\bar{\underset{\sim}{u}}\}\,d\underset{\sim}{Y}_o(\underset{\sim}{k},\omega) - d\underset{\sim}{Y}_o(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\nabla}\bar{\underset{\sim}{u}}$$

$$= i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{B}})\,d\underset{\sim}{Z}(\underset{\sim}{k},\omega) - i\{\underset{\sim}{k}\cdot d\underset{\sim}{Z}(\underset{\sim}{k},\omega)\}\,\bar{\underset{\sim}{B}}$$

$$- d\underset{\sim}{Z}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{\nabla}\bar{\underset{\sim}{B}} \qquad (2.50)$$

clearly indicating that $d\underset{\sim}{Y}_o$ is really of the form

$$d\underset{\sim}{Y}_o = d\underset{\sim}{Y}_o(\underset{\sim}{k},\omega;\underset{\sim}{x},t) \qquad (2.51)$$

with spatial and temporal variation on the length and time scales of the mean fields. It is also to be expected that $d\underset{\sim}{Z}$ will vary on these scales. Thus the fluctuating fields $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ will in general be *nonstationary, inhomogeneous random functions*. In Chapter 4 a method for treating fields of this nature in a more detailed fashion will be described. However, for the present we shall assume that *the length and time scales of the fluctuating fields are sufficiently short compared to the length and time scales of the mean fields for* $\underset{\sim}{u}'$ *to be treated as a stationary, homogeneous random function, to a reasonable approximation.* This assumption is the underlying feature of the expansion technique

developed by *Rädler (1968a)* and others for evaluating the term $\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}$ *(see P.H. Roberts, 1971a; Krause and Rädler, 1971; Moffatt, 1970a)*.

Neglecting spatial gradients of $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$, we find that (2.50) reduces to the form (2.45), with

$$\begin{aligned} \hat{\overline{\underset{\sim}{u}}}(0,0) &\to \overline{\underset{\sim}{u}} \\ \hat{\overline{\underset{\sim}{B}}}(0,0) &\to \overline{\underset{\sim}{B}} \end{aligned} \qquad (2.52)$$

The result (2.49) then follows directly from the assumption that $\underset{\sim}{u}'$ is stationary and homogeneous.

2.4.5 Solution of the fluctuating induction equation for "wave" mean fields

The assumption that $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$ are of the form implied by (2.40), (2.41), (2.43), and (2.44) is clearly a gross over-simplification. Among other things, it implies that the energy stored in the mean fields is not finite, and that these fields over all space and time. The assumption is thus strictly applicable only to the unrealistic case of *turbulent motion in an infinite fluid*, with no boundary conditions implied on $\overline{\underset{\sim}{u}}$ or $\overline{\underset{\sim}{B}}$.

If we make the more realistic assumption that the fields $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$ go to zero as r^{-3} in the limit as $r \to \infty$, the behaviour of the mean field transforms near $\underset{\sim}{K} = 0$ must be *(Phillips, 1956)*

$$\hat{\underset{\sim}{F}}(\underset{\sim}{K},\Omega) = \left\{ \underset{\approx}{I} - \underset{\sim}{K}\underset{\sim}{K}/K^2 \right\} \cdot \hat{\underset{\sim}{F}}(0,0)\, \delta(\Omega) + O(K) \tag{2.53}$$

and (2.45) must be replaced by a much more complicated equation.

On the other hand, if we consider the case of *wave mean fields* of the type

$$\hat{\underset{\sim}{u}}(\underset{\sim}{K},\Omega) = \hat{\underset{\sim}{u}}(\underset{\sim}{K}_1,\Omega_1)\ \delta(\underset{\sim}{K}-\underset{\sim}{K}_1)\ \delta(\Omega-\Omega_1) \tag{2.54}$$

$$\hat{\underset{\sim}{B}}(\underset{\sim}{K},\Omega) = \hat{\underset{\sim}{B}}(\underset{\sim}{K}_o,\Omega_o)\ \delta(\underset{\sim}{K}-\underset{\sim}{K}_o)\ \delta(\Omega-\Omega_o) \tag{2.55}$$

where

$$\underset{\sim}{K}_o \cdot \hat{\underset{\sim}{B}}(\underset{\sim}{K}_o,\Omega) = 0 \tag{2.56}$$

equation (2.42) becomes

$$\begin{aligned} \{i\omega + \eta k^2\}\, d\underset{\sim}{Y}_o(\underset{\sim}{k},\omega) & \\ - i\,\underset{\sim}{k} \times \left\{ \hat{\underset{\sim}{u}}(\underset{\sim}{K}_1,\Omega_1) \times d\underset{\sim}{Y}_o(\underset{\sim}{k}-\underset{\sim}{K}_1,\omega-\Omega_1) \right\} & \\ = i\underset{\sim}{k} \times \left\{ d\underset{\sim}{Z}(\underset{\sim}{k}-\underset{\sim}{K}_o,\omega-\Omega_o) \times \hat{\underset{\sim}{B}}(\underset{\sim}{K}_o,\Omega_o) \right\} & \end{aligned} \tag{2.57}$$

Here again we meet with the problem of infinite energy in the mean fields, but the spatial and temporal gradients of these fields are non-zero.

2.5 Evaluation of $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$

2.5.1 The *Rädler* expansion technique

In the *Rädler expansion technique (Rädler, 1968a; see Krause and Rädler, 1971)* $\overline{\underset{\sim}{B}}$ is expanded in the second integral on the right hand side of equation (2.22), according to

$$\overline{\underset{\sim}{B}}(\underset{\sim}{x}+\underset{\sim}{r}, t+\tau) = \overline{\underset{\sim}{B}}(\underset{\sim}{x},t) + \underset{\sim}{r}\cdot\underset{\sim}{\nabla}\,\overline{\underset{\sim}{B}}(\underset{\sim}{x},t) + \tau\frac{\partial}{\partial t}\overline{\underset{\sim}{B}}(\underset{\sim}{x},t) + \ldots \tag{2.58}$$

The expression for $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}$ becomes

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}\}_i = \sum_{\kappa,\nu} g^{(\kappa,\nu)}_{iqn_1\cdots n_\kappa} \frac{\partial^{\kappa+\nu}\overline{B}_q(\underset{\sim}{x},t)}{\partial x_{n_1}\cdots\partial x_{n_\kappa}\,\partial t^\nu} + \epsilon_{ijk}\int_V d\underset{\sim}{x}'\, G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}',t_o)\,\overline{u_j'(\underset{\sim}{x},t)\,B_{o\ell}'(\underset{\sim}{x}',t_o)} \tag{2.59}$$

where

$$g^{(\kappa,\nu)}_{iqn_1\cdots n_\kappa} = -\epsilon_{ijk}\,\epsilon_{\ell mn}\,\epsilon_{npq}\cdot\frac{(-1)^{\kappa+\nu}}{\kappa!\,\nu!}\cdot \int_V d\underset{\sim}{r}\int_0^{t-t_o} d\tau\; G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}-\underset{\sim}{r},t-\tau)\cdot \frac{\partial}{\partial r^m}\left\{R_{jp}(\underset{\sim}{x},t;\underset{\sim}{x}+\underset{\sim}{r},t-\tau)\,r_{n_1}r_{n_2}\cdots r_{n_\kappa}\tau^\nu\right\} \tag{2.60}$$

The coefficients (2.60) are then evaluated under a number of assumptions concerning:

a. the nature of the Green's dyadic G_{ij} ;

b. the symmetry properties of the turbulence, $\underset{\sim}{u}'$;

c. the nature of the mean flow, $\overline{\underset{\sim}{u}}$.

Once $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'_0}$ has been evaluated, higher approximations to $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ can be obtained from (2.26)-(2.30). The higher-order terms in (2.28) will involve higher-order velocity correlations. For example, $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'_1}$ involves *three-point, three-time* correlations, $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'_2}$ involves *four-point, four-time* correlations, and so on.

2.5.2 Choice of Green's function

The *Green's dyadic* most commonly used is that appropriate to an infinite domain for the case $\overline{\underset{\sim}{u}} \equiv 0$.

$$G_{ij}(\underset{\sim}{x},t\,;\underset{\sim}{x}-\underset{\sim}{r},t-\tau) = \delta_{ij}\,G(r,\tau)$$

$$G(r,\tau) = (4\pi\eta\tau)^{-3/2}\,e^{-r^2/4\eta\tau}\;,\quad \tau \geq 0 \qquad (2.61)$$

$$= 0\;,\quad \tau < 0$$

The choice of an *infinite domain* Green's function is justified on the grounds that the turbulence extends over a region large compared with the turbulence *correlation length*.

Results obtained using (2.61) can be extended to the case $\overline{\underset{\sim}{u}}$ = constant by transforming to a frame in uniform relative motion *(Krause and Rädler, 1971)*. Green's functions appropriate to several other types of mean flow have also been studied *(Krause and Rädler, 1971, pp. 36-37; Rädler, 1964, 1969a; Krause, 1968a,b)*.

2.5.3 Comparison of expansion and Fourier transform techniques

The zero-order term obtained from (2.59) may be compared with (2.49) when $\overline{\underset{\sim}{u}} = 0$ and G_{ij} is given by (2.61). We have, from (2.59),

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'_o}\}_i \approx g_{iq}^{(0,0)} \overline{B}_q + \epsilon_{ijk} \int_V d\underset{\sim}{x}' \, G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}',t_o) \, \overline{u_j'(\underset{\sim}{x},t) B'_{o\ell}(\underset{\sim}{x}',t_o)}$$

$$\approx \epsilon_{ij\ell}\, \epsilon_{\ell mn}\, \epsilon_{npq} \int_{\substack{\text{all}\\ \text{space}}} d\underset{\sim}{r} \int_0^{t-t_o} d\tau \, (4\pi\eta\tau)^{-3/2} \, e^{-r^2/4\eta\tau} \; .$$

$$\cdot \iint_{\underset{\sim}{k}\,\omega} i k_m \, \Phi_{jp}(\underset{\sim}{k},\omega) \, e^{-i\{\underset{\sim}{k}\cdot\underset{\sim}{r} + \omega\tau\}} \, d\underset{\sim}{k}\, d\omega$$

$$+ \epsilon_{ij\ell} \int_{\substack{\text{all}\\ \text{space}}} d\underset{\sim}{r} \, \{4\pi\eta(t-t_o)\}^{-3/2} \, e^{-r^2/4\eta(t-t_o)} \; . \qquad (2.62)$$

$$\cdot \iiiint_{\underset{\sim}{k}\,\omega\,\underset{\sim}{k}'\omega'} \overline{dZ_j^*(\underset{\sim}{k},\omega)\, dY_{o\ell}(\underset{\sim}{k}',\omega')} \, e^{i\{\underset{\sim}{k}'\cdot\underset{\sim}{x}' - \underset{\sim}{k}\cdot\underset{\sim}{x} + \omega' t_o - \omega t\}}$$

where we have made use of the definitions

$$R_{ij}(\underset{\sim}{x},t\,;\underset{\sim}{x}+\underset{\sim}{r},t+\tau) = \iint_{\underset{\sim}{k}\,\omega} \Phi_{ij}(\underset{\sim}{k},\omega) \, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{r} + \omega\tau\}} \, d\underset{\sim}{k}\, d\omega \qquad (2.63)$$

and (2.35)-(2.38). (2.62) may be simplified by using the identities

$$\int_{\text{all space}} d\underset{\sim}{r} \; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r} - r^2/4\eta\tau} \equiv (4\pi\eta\tau)^{3/2} \, e^{-\eta k^2\tau} \qquad (2.64a)$$

$$\int_0^{t-t_o} d\tau \; e^{-\{i\omega + \eta k^2\}\tau} \equiv (i\omega + \eta k^2)^{-1} \{1 - e^{-(i\omega+\eta k^2)(t-t_o)}\} \qquad (2.64b)$$

$$\int_{\text{all space}} d\underset{\sim}{r} \int_0^{t-t_0} d\tau \; (4\pi\eta\tau)^{3/2} \, e^{-r^2/4\eta\tau - i(\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega\tau)}$$

$$\equiv (i\omega+\eta k^2)^{-1}\left\{1 - e^{-(i\omega+\eta k^2)(t-t_0)}\right\} \qquad (2.64c)$$

From (2.62), (2.64), and (2.46),

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_0'}\}_i \approx \epsilon_{ijk}\,\epsilon_{kmn}\,\epsilon_{npq} \iint_{\underset{\sim}{k}\omega} \frac{ik_m\,\Phi_{jp}(\underset{\sim}{k},\omega)}{(i\omega+\eta k^2)}\, d\underset{\sim}{k}\, d\omega \cdot \overline{B}_q$$

$$+ \,\epsilon_{ij\ell} \iiint_{\underset{\sim}{k}\omega\underset{\sim}{k}'} \overline{dZ_j^*(\underset{\sim}{k},\omega)\, dY_{0\ell}'(\underset{\sim}{k}')}\; e^{i(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} - (\eta k'^2 + i\omega)t}$$

$$(2.65)$$

Equation (2.65) is clearly identical to (2.49), with $\hat{\underset{\sim}{B}}(0,0)$ replaced by $\overline{\underset{\sim}{B}}$, and $\hat{\underset{\sim}{u}}(0,0) \equiv 0$. This equivalence shows how the *uniform, time-independent field* assumption in the Fourier transform solution corresponds to the neglect of space and time derivatives in the Rädler expansion technique, when the infinite-domain Green's function (2.61) is used.

2.5.4 First order smoothing, and scaling of terms in the induction equation

In most studies to date, it has been assumed that the neglect of $\underset{\sim}{C}'$ *(see equation 2.12)* in the fluctuating induction equation (2.10) is a valid approximation. In other words, it has been assumed that

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'} = \overline{\underset{\sim}{u}' \times \underset{\sim}{B}_0'} \qquad (2.66)$$

Lerche (1971a) has named (2.66) *the first order smoothing approximation*. The validity of this approximation may be investigated by considering the scaling of terms in the fluctuating part of the induction equation.

Assume that the fluctuating fields $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ vary on the scale of the turbulence *correlation length*, λ_c and *correlation time*, τ_c, while the mean fields $\bar{\underset{\sim}{u}}$ and $\bar{\underset{\sim}{B}}$ vary on the scales L and T. Further, assume that the magnitudes of $\bar{\underset{\sim}{u}}$, $\underset{\sim}{u}'$, $\bar{\underset{\sim}{B}}$, and $\underset{\sim}{B}'$ are $\bar{\mathcal{U}}$, $\mathcal{U}'$, $\bar{\mathcal{B}}$, and $\mathcal{B}'$. Then equation (2.10), which will be rewritten here for convenience,

$$\underset{①}{\partial\underset{\sim}{B}'/\partial t} \; \underset{②}{-\eta\nabla^2\underset{\sim}{B}'} \; \underset{③}{-\operatorname{curl}\{\bar{\underset{\sim}{u}}\times\underset{\sim}{B}'\}} \; \underset{④}{-\operatorname{curl}\{\underset{\sim}{u}'\times\bar{\underset{\sim}{B}}\}}$$

$$\underset{⑤}{-\operatorname{curl}\{\underset{\sim}{u}'\times\underset{\sim}{B}'-\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}\}} = 0$$

scales as

$$\frac{\mathcal{B}'\eta}{\lambda_c^2}\left\{ \underset{①}{q} + \underset{②}{1} + R_m'\left(1+\frac{\lambda_c}{L}\right)\left(\underset{③}{\frac{\bar{\mathcal{U}}}{\mathcal{U}'}} + \underset{④}{\frac{\bar{\mathcal{B}}}{\mathcal{B}'}} + \underset{⑤}{1}\right)\right\} \sim 0 \qquad (2.67)$$

where

$$R_m' \equiv \mathcal{U}'\lambda_c/\eta \qquad (2.68)$$

and

$$q \equiv \lambda_c^2/\eta\tau_c \qquad (2.69)$$

When $\underset{\sim}{\bar{u}} = 0$, the term ③ in (2.67) drops out, and the equation reduces to

$$\frac{\mathcal{B}'\eta}{\lambda_c^2}\left\{ \underset{①}{q} + \underset{②}{1} + \underset{④}{R_m'(1+\lambda_c/L)(\bar{\mathcal{B}}/\mathcal{B}'} + \underset{⑤}{1)} \right\} \sim 0 \tag{2.70}$$

If $\lambda_c \ll L$, we have

$$\frac{\mathcal{B}'\eta}{\lambda_c^2}\left\{ \underset{①}{q} + \underset{②}{1} + \underset{④}{R_m'(\bar{\mathcal{B}}/\mathcal{B}'} + \underset{⑤}{1)} \right\} \sim 0 \tag{2.70'}$$

First order smoothing corresponds to the neglect of term ⑤ in (2.67)-(2.70'). There are two limiting cases to be considered *(Krause and Rädler, 1971)*. In the first limiting case, $q \ll 1$. The neglect of ⑤ then implies that

$$\mathcal{B}'/\bar{\mathcal{B}} \sim R_m' \qquad (q \ll 1) \tag{2.71}$$

and the condition for consistency is

$$R_m' \ll 1 \qquad (q \ll 1) \tag{2.71'}$$

In the second limiting case, $q \gg 1$. The neglect of ⑤ then implies that

$$\mathcal{B}'/\bar{\mathcal{B}} \sim R_m'/q \qquad (q \gg 1) \tag{2.72}$$

and the condition for consistency is

$$R_m' \ll q \qquad (q \gg 1) \tag{2.72'}$$

It should be noted that in deriving (2.72') we have assumed that diffusion is negligible in the fluctuating induction equation. However, a dynamo cannot operate without diffusion *(Cowling, 1957, 1965)*. It follows that if (2.72') is to be valid, diffusion must be significant in the mean field induction equation. If this is not the case, term 2 must be retained in (2.70'), and the condition for 5 to be negligible is again $R'_m \ll 1$. Bearing this restriction in mind, we may combine the consistency conditions (2.71') and (2.72'), and write

$$R'_m \ll (1 + q) \tag{2.72''}$$

The consistency check carried out above can be extended to include the mean field induction equation (2.9). This equation scales as

$$\underset{1}{\partial \bar{\underset{\sim}{B}}/\partial t} - \underset{2}{\eta \nabla^2 \bar{\underset{\sim}{B}}} - \underset{3}{\mathrm{curl}(\bar{\underset{\sim}{u}} \times \bar{\underset{\sim}{B}})} - \underset{4}{\mathrm{curl}(\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'})} = 0$$

$$\frac{\bar{\mathcal{B}}\eta}{L^2}\Big\{ \underset{1}{Q} + \underset{2}{1} + R'_m \frac{L}{\lambda_c}\big(\underset{3}{\bar{\mathcal{U}}/\mathcal{U}'} + \underset{4}{\mathcal{B}'/\bar{\mathcal{B}}} \big) \Big\} \sim 0 \tag{2.73}$$

where

$$Q \equiv L^2/\eta T \tag{2.74}$$

When $\bar{u} = 0$, (2.73) reduces to the condition

$$\mathcal{B}'/\bar{\mathcal{B}} \sim \frac{1}{R'_m}\frac{\lambda_c}{L}(1 + Q) \tag{2.75}$$

Combining (2.75) with (2.71)-(2.72'), we have

$$(q \ll 1) \qquad \mathcal{B}'/\bar{\mathcal{B}} \sim R_m' \sim (\lambda_c/L)^{1/2}(1+Q)^{1/2} \ll 1 \qquad (2.76)$$

$$(q \gg 1) \qquad \mathcal{B}'/\bar{\mathcal{B}} \sim R_m'/q \sim (\lambda_c/qL)^{1/2}(1+Q)^{1/2} \ll 1 \qquad (2.76')$$

These conditions are clearly compatible with the assumption that $\lambda_c \ll L$, provided that Q is not too large.

2.6 Solution of the mean field induction equation

2.6.1 General considerations

When the source term $\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}$ has been evaluated, it must be substituted into the mean field induction equation (2.9), and (2.9) must then be solved for $\underset{\sim}{\overline{B}}$. At this stage any assumptions like (2.52) about "uniformity" and "time-independence" of the mean fields are dropped, and $\underset{\sim}{\overline{u}}$ and $\underset{\sim}{\overline{B}}$ are allowed to vary in the expression for $\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}$. Ideally, the coefficients in an expansion like (2.59) should be allowed to vary as well, to represent any inhomogeneity of $\underset{\sim}{u}'$ required by the boundary conditions or by the Navier-Stokes equation.

2.6.2 The mean field dispersion relation

In certain cases, it is possible to keep the solution of the mean field equation consistent with the assumptions made in deriving the expression for $\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}$. For example, if $\underset{\sim}{\overline{u}} = 0$ and $\underset{\sim}{\overline{B}}$ has the *wave-like* form implied by (2.41) and (2.55), it is possible to derive a *dispersion relation* for the mean field. This problem has been considered by *Lerche (1971a,b)*, *Gilliland and Aldridge (1973)*, and *Krause and Roberts (1973)*.

Substituting (2.41) and (2.55) into (2.9), and making use of the *first order smoothing approximation* (2.66),

$$(i\Omega + \eta K^2)\,\hat{\hat{\underset{\sim}{B}}}(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot x + \Omega t\}}$$

$$= \text{curl}\,\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_0'}\} \tag{2.77}$$

From (2.57), with $\hat{\hat{\underset{\sim}{u}}}(\underset{\sim}{K}_1, \Omega_1) = 0$,

$$d\underset{\sim}{Y}_0(\underset{\sim}{k},\omega) = \{i\omega + \eta k^2\}^{-1}\{d\underset{\sim}{Z}(\underset{\sim}{k}-\underset{\sim}{K}, \omega-\Omega) \times \hat{\hat{\underset{\sim}{B}}}(\underset{\sim}{K},\Omega)\}$$

$$+ d\underset{\sim}{Y}_0'(\underset{\sim}{k})\, \delta(\omega - i\eta k^2) \tag{2.78}$$

so that

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_0'}\}_i = i\epsilon_{ij\ell}\,\epsilon_{\ell mn}\,\epsilon_{npq}\,\hat{\hat{B}}_q(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x} + \Omega t\}}\, .$$

$$\cdot \iint_{\underset{\sim}{k}\,\omega} \frac{(\underset{\sim}{k}+\underset{\sim}{K})_m\, \Phi_{jp}(\underset{\sim}{k},\omega)}{i(\omega+\Omega) + \eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\, d\underset{\sim}{k}\, d\omega$$

$$+ i\epsilon_{ij\ell} \iiint_{\underset{\sim}{k}\,\omega\,\underset{\sim}{k}'} \overline{dZ_j^*(\underset{\sim}{k},\omega)\, dY_{0\ell}'(\underset{\sim}{k}')}\, e^{i(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} - (\eta k'^2 - i\omega)t} \tag{2.79}$$

Substituting (2.79) into (2.77),

$$\Big\{(i\Omega + \eta K^2)\,\delta_{rq} + \epsilon_{rsi}\,\epsilon_{ij\ell}\,\epsilon_{\ell mn}\,\epsilon_{npq}\, K_s\, \cdot$$

$$\cdot \iint_{\underset{\sim}{k}\,\omega} \frac{(\underset{\sim}{k}+\underset{\sim}{K})_m\, \Phi_{jp}(\underset{\sim}{k},\omega)}{i(\omega+\Omega) + \eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\, d\underset{\sim}{k}\, d\omega \Big\}\, \hat{\hat{B}}_q\, e^{i(\underset{\sim}{K}\cdot\underset{\sim}{x} + \Omega t)} \tag{2.80}$$

$$= -\epsilon_{rsi}\,\epsilon_{ij\ell} \iiint_{\underset{\sim}{k}\,\omega\,\underset{\sim}{k}'} (\underset{\sim}{k}'-\underset{\sim}{k})_s\, \overline{dZ_j^*(\underset{\sim}{k},\omega)\, dY_{0\ell}'(\underset{\sim}{k}')}\, e^{i(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} - (\eta k'^2 - i\omega)t}$$

The expression on the right hand side of (2.80) is clearly a decaying function of time, while the expression on the left hand side may grow with time ($\text{Im}\ \Omega < 0$) or decay with time ($\text{Im}\ \Omega > 0$). If $\text{Im}\ \Omega < 0$, after a sufficiently long

time the right hand side of (2.80) will become negligible in comparison to the left hand side, and the form of the mean field $\bar{\underset{\sim}{B}}$ will be determined by the *dispersion relation*

$$\Big\{(i\Omega+\eta K^2)\delta_{rq} + \epsilon_{rsi}\epsilon_{ij\ell}\epsilon_{\ell mn}\epsilon_{npq} K_s \cdot \tag{2.81}$$

$$\cdot \iint_{\underset{\sim}{k}\omega} \frac{(\underset{\sim}{k}+\underset{\sim}{K})_m \Phi_{jp}(\underset{\sim}{k},\omega)}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2} d\underset{\sim}{k}\, d\omega \Big\} \hat{\bar{B}}_q(\underset{\sim}{K},\Omega) = 0$$

Making use of (2.56) and the identities

$$\epsilon_{\ell mn}\epsilon_{npq} = \delta_{\ell p}\delta_{mq} - \delta_{\ell q}\delta_{mp} \tag{2.82a}$$

$$\epsilon_{ij\ell}\epsilon_{\ell mn}\epsilon_{npq} = \epsilon_{ijp}\delta_{mq} - \epsilon_{ijq}\delta_{mp} \tag{2.82b}$$

$$\epsilon_{rsi}\epsilon_{ij\ell}\epsilon_{\ell mn}\epsilon_{npq} = \delta_{rj}\delta_{sp}\delta_{mq} - \delta_{rp}\delta_{sj}\delta_{mq} - \delta_{rj}\delta_{sq}\delta_{mp} + \delta_{rq}\delta_{sj}\delta_{mp} \tag{2.82c}$$

we find that (2.81) reduces to

$$\Big\{ i\Omega + \eta K^2 + \iint_{\underset{\sim}{k}\omega} \frac{K_j \Phi_{jp}(\underset{\sim}{k},\omega)(\underset{\sim}{k}+\underset{\sim}{K})_p}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2} d\underset{\sim}{k}\, d\omega \Big\} \hat{\bar{B}}_r(\underset{\sim}{K},\Omega)$$

$$+ \Big\{ K_s \iint_{\underset{\sim}{k}\omega} \frac{k_q[\Phi_{rs}(\underset{\sim}{k},\omega) - \Phi_{sr}(\underset{\sim}{k},\omega)]}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2} d\underset{\sim}{k}\, d\omega \Big\} \hat{\bar{B}}_q(\underset{\sim}{K},\Omega)$$

$$= 0 \tag{2.83}$$

Separating the real and imaginary parts of (2.83), we may write the equation in block matrix form.

$$\begin{pmatrix} \underset{\approx}{a} & -\underset{\approx}{b} \\ \underset{\approx}{b} & \underset{\approx}{a} \end{pmatrix} \begin{pmatrix} \mathcal{R}e\, \hat{\underset{\sim}{\hat{B}}} \\ \mathcal{I}m\, \hat{\underset{\sim}{\hat{B}}} \end{pmatrix} = 0 \tag{2.84}$$

where

$$\underset{\approx}{a} \equiv \mathcal{R}e\, \underset{\approx}{I}^{(2)} - \{ \mathcal{I}m\,\Omega - \eta K^2 - \mathcal{R}e\, I^{(1)} \} \underset{\approx}{I} \tag{2.85}$$

$$\underset{\approx}{b} \equiv \mathcal{I}m\, \underset{\approx}{I}^{(2)} + \{ \mathcal{R}e\,\Omega + \mathcal{I}m\, I^{(1)} \} \underset{\approx}{I} \tag{2.86}$$

with

$$\hat{\underset{\sim}{\hat{B}}}(\underset{\sim}{K},\Omega) \equiv \mathcal{R}e\, \hat{\underset{\sim}{\hat{B}}}(\underset{\sim}{K},\Omega) + i\, \mathcal{I}m\, \hat{\underset{\sim}{\hat{B}}}(\underset{\sim}{K},\Omega) \tag{2.87}$$

$$I^{(1)}(\underset{\sim}{K},\Omega) \equiv \iint_{\underset{\sim}{k}\omega} \frac{K_j \Phi_{jp}(\underset{\sim}{k},\omega)(\underset{\sim}{k}+\underset{\sim}{K})_p}{i(\omega+\Omega) + \eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\, d\underset{\sim}{k}\, d\omega \tag{2.88}$$

$$I^{(2)}_{ij}(\underset{\sim}{K},\Omega) \equiv \iint_{\underset{\sim}{k}\omega} \frac{K_\ell k_j \{ \Phi_{i\ell}(\underset{\sim}{k},\omega) - \Phi_{\ell i}(\underset{\sim}{k},\omega)\}}{i(\omega+\Omega) + \eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\, d\underset{\sim}{k}\, d\omega \tag{2.89}$$

$$= 2i K_\ell \iint_{\underset{\sim}{k}\omega} \frac{k_j\, \mathcal{I}m\, \Phi_{i\ell}(\underset{\sim}{k},\omega)}{i(\omega+\Omega) + \eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\, d\underset{\sim}{k}\, d\omega \tag{2.89'}$$

(2.89') follows from (2.89) since Φ_{ij} must satisfy the condition of *Hermitian symmetry*

$$\Phi_{ij}(\underset{\sim}{k},\omega) = \Phi^{*}_{ji}(\underset{\sim}{k},\omega) \tag{2.89"}$$

as a consequence of (2.4') and the equation of the type (2.34') which relates Φ_{ij} to R_{ij} .

Equation (2.84) has a non-trivial solution only if

$$\det \begin{pmatrix} \underset{\approx}{a} & -\underset{\approx}{b} \\ \underset{\approx}{b} & \underset{\approx}{a} \end{pmatrix} = 0 \qquad (2.90)$$

Because of the form of the block matrix, (2.90) may be rewritten in the equivalent form

$$\det{}^2 \begin{pmatrix} \underset{\approx}{a} & -\underset{\approx}{b} \\ \underset{\approx}{b} & \underset{\approx}{a} \end{pmatrix} = \det \begin{pmatrix} \underset{\approx}{a} & -\underset{\approx}{b} \\ \underset{\approx}{b} & \underset{\approx}{a} \end{pmatrix} \det \begin{pmatrix} \underset{\approx}{a} & \underset{\approx}{b} \\ -\underset{\approx}{b} & \underset{\approx}{a} \end{pmatrix}$$

$$= \det \begin{pmatrix} [\underset{\approx}{a}\underset{\approx}{a} + \underset{\approx}{b}\underset{\approx}{b}] & [\underset{\approx}{a}\underset{\approx}{b} - \underset{\approx}{b}\underset{\approx}{a}] \\ [\underset{\approx}{b}\underset{\approx}{a} - \underset{\approx}{a}\underset{\approx}{b}] & [\underset{\approx}{a}\underset{\approx}{a} + \underset{\approx}{b}\underset{\approx}{b}] \end{pmatrix}$$

$$= \det \begin{pmatrix} [\underset{\approx}{a}\underset{\approx}{a} + \underset{\approx}{b}\underset{\approx}{b}] & [\mathcal{R}e\,\underset{\approx}{I}^{(2)}\,\mathcal{I}m\,\underset{\approx}{I}^{(2)} - \mathcal{I}m\,\underset{\approx}{I}^{(2)}\,\mathcal{R}e\,\underset{\approx}{I}^{(2)}] \\ [\mathcal{I}m\,\underset{\approx}{I}^{(2)}\,\mathcal{R}e\,\underset{\approx}{I}^{(2)} - \mathcal{R}e\,\underset{\approx}{I}^{(2)}\,\mathcal{I}m\,\underset{\approx}{I}^{(2)}] & [\underset{\approx}{a}\underset{\approx}{a} + \underset{\approx}{b}\underset{\approx}{b}] \end{pmatrix}$$

$$= 0 \qquad (2.91)$$

2.6.3 An alternative derivation of the mean field dispersion relation

The results (2.84)-(2.91) may be compared with the results obtained using the Rädler approximation technique in the first order smoothing approximation. The assumption that the initial conditions have had time to "die out" is equivalent to ignoring the second term in (2.59) and setting the upper limit of the time integration in (2.60) equal to infinity. Then, making use of (2.61) and (2.63),

$$\{\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'_o}\}_i = \sum_{\kappa,\nu} g^{(\kappa,\nu)}_{iqn_1\cdots n_\kappa} \frac{\partial^{\kappa+\nu}}{\partial x_{n_1}\cdots\partial x_{n_\kappa}\partial t^\nu} \overline{B}_q(\underset{\sim}{x},t) \tag{2.92}$$

where

$$g^{(\kappa,\nu)}_{iqn_1\cdots n_\kappa} = \frac{(-1)^{\kappa+\nu+1}}{\kappa!\,\nu!} \epsilon_{ij\ell}\epsilon_{\ell mn}\epsilon_{npq} \cdot$$

$$\cdot \iint_{\underset{\sim}{k}\,\omega} \Phi_{jp}(\underset{\sim}{k},\omega)\, d\underset{\sim}{k}\, d\omega \int_{\text{all space}} d\underset{\sim}{r} \int_0^\infty d\tau\, (4\pi\eta\tau)^{-3/2} e^{-r^2/4\eta\tau} \cdot$$

$$\cdot \frac{\partial}{\partial r_m}\left\{ r_{n_1}\cdots r_{n_\kappa} \tau^\nu e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega\tau)}\right\} \tag{2.93}$$

The assumption, used in deriving (2.92), that term-by-term integration is valid in (2.22) implies that the series on the right hand side of (2.92) may be Fourier transformed term by term to give a convergent Fourier representation *(see, for example, Whittaker and Watson, 1927, p. 78).* Thus, making use of (2.41),

$$\{\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'_o}\}_i = \iint_{\underset{\sim}{K}\,\Omega} d\underset{\sim}{K}\, d\Omega\, e^{i\{\underset{\sim}{K}\cdot x+\Omega t\}}\, \hat{\overline{B}}_q(\underset{\sim}{K},\Omega) \cdot$$

$$\cdot \sum_{\kappa,\nu} g^{(\kappa,\nu)}_{iqn_1\cdots n_\kappa}\, i^{\kappa+\nu}\, \Omega^\nu K_{n_1}\cdots K_{n_\kappa}$$

so that, taking the Fourier transform,

$$\widehat{\{\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'_o}\}}_i = \hat{\overline{B}}_q(\underset{\sim}{K},\Omega) \sum_{\kappa,\nu} g^{(\kappa,\nu)}_{iqn_1\cdots n_\kappa}\, i^{\kappa+\nu}\Omega^\nu K_{n_1}\cdots K_{n_\kappa} \tag{2.94}$$

The Fourier transform of the mean field induction equation (2.9) is then

$$\{i\Omega+\eta K^2\}\,\hat{\hat{B}}_r = i\,\epsilon_{rsi}\,K_s\,\{\widehat{\underset{\sim}{u}'\times\underset{\sim}{B}_o'}\}_i$$

$$= i\,\epsilon_{rsi}\,K_s\,\hat{\hat{B}}_q \sum_{\kappa,\nu} g^{(\kappa,\nu)}_{iqn_1\ldots n_\kappa}\, i^{\kappa+\nu}\,\Omega^\nu\, K_{n_1}\ldots K_{n_\kappa} \qquad (2.95)$$

$$\approx i\,\epsilon_{rsi}\,K_s\,\hat{\hat{B}}_q \{ g^{(0,0)}_{iq} + i\,g^{(1,0)}_{iqn_1}K_{n_1} + i\,g^{(0,1)}_{iq}\,\Omega - g^{(2,0)}_{iqn_1n_2}K_{n_1}K_{n_2} - g^{(1,1)}_{iqn_1}\,\Omega K_{n_1} - g^{(0,2)}_{iq}\,\Omega^2 + \ldots \} \qquad (2.95')$$

retaining only the first few terms on the right hand side.

Taking real and imaginary parts of (2.95'), and noting that the $g^{(\kappa,\nu)}_{iqn_1\ldots n_\kappa}$ are purely real, by definition, we obtain an equation equivalent to (2.84). In this equation, $I^{(1)}$ and $\underset{\approx}{I}^{(2)}$ are expressed as power series expansions of (2.88) and (2.89) about $(\underset{\sim}{K},\Omega) = (0,0)$, integrated term by term.

2.6.4 Symmetry considerations

If the spectrum tensor of the turbulence is symmetric under interchange of indices,

$$\Phi_{ij}(\underset{\sim}{k},\omega) = \Phi_{ji}(\underset{\sim}{k},\omega) \qquad (2.96)$$

(i.e. Φ_{ij} is **purely** real, because of *equation 2.89"*), then $\underset{\approx}{I}^{(2)}$ vanishes identically, and (2.85)-(2.91) reduce to the equations

$$\mathfrak{Im}\,\Omega - \eta K^2 - \mathfrak{Re}\,I^{(1)} = 0 \tag{2.97}$$

$$\mathfrak{Re}\,\Omega + \mathfrak{Im}\,I^{(1)} = 0 \tag{2.98}$$

We shall consider these equations in detail in *Chapter 3*.

2.7 The mean field hydromagnetic dynamo problem

The techniques of mean field electrodynamics which have been outlined in this chapter can also be applied to the *hydromagnetic dynamo problem (Moffatt, 1972)*. We shall examine this application in *Chapter 6*.

2.8 Summary of Chapter 2

This chapter is concerned with *mean field electrodynamics* and its application to the kinematic dynamo problem. Most of the material presented is taken from the recent literature on the subject; however, several original contributions appear.

In *section 2.2* a new terminology, related to that used in other disciplines, is proposed for several types of *stationary, homogeneous turbulence* with particular invariance properties.

In *section 2.5* a comparison is presented of the results obtained using *Fourier transform* and *Green's function* techniques in the mean field electrodynamic approach to the dynamo problem.

Finally, in *section 2.6* the *mean field dispersion relation* for *wave* mean fields is cast in a novel determinantal form.

3. THE KINEMATIC DYNAMO PROBLEM AND PT-INVARIANT TURBULENCE

3.1 Helicity, and the maintenance of nearly uniform, nearly time-independent mean fields

As noted in section 1.4.1, a solution to the kinematic dynamo problem consists of a pair of fields ($\underset{\sim}{u}$, $\underset{\sim}{B}$) such that $\underset{\sim}{u}$ is "allowable", $\underset{\sim}{B}$ satisfies the induction equation, and the mean magnetic energy stored in the conducting fluid grows with time or remains constant. In the MFE approach, it is the energy associated with the mean field $\overline{\underset{\sim}{B}}$ which is required to grow (or remain constant), and the question to be asked is: what restrictions must be placed on the fluctuating velocity field $\underset{\sim}{u}'$ to ensure that dynamo action will occur? An important restriction of this type has been proposed by *Krause (1968a)*, *Rädler (1968a)*, and *Moffatt (1970a)*, who point out that perhaps the simplest method of ensuring that dynamo action *can* occur when $\overline{\underset{\sim}{u}} = 0$ is to require that

$$\overline{\underset{\sim}{u}' \cdot \text{curl } \underset{\sim}{u}'} \neq 0 \qquad (3.1)$$

This quantity has been given the name *helicity* by *Moffatt (1969)*. Its effect is seen most clearly by making use of the Fourier-Stieltjes representation (2.35) for $\underset{\sim}{u}'$ to obtain *(Moffatt, 1970a)*

$$\overline{\underset{\sim}{u}' \cdot \text{curl}\, \underset{\sim}{u}'} = i\epsilon_{ij\ell} \iint_{\underset{\sim}{k}\,\omega} k_i\, \Phi_{j\ell}(\underset{\sim}{k},\omega)\, d\underset{\sim}{k}\, d\omega \qquad (3.2)$$

If the mean field $\overline{\underset{\sim}{B}}$ is virtually uniform and time-independent on the length and time scales of the turbulence, we have from (2.65) that

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}\}_i \approx i\,\epsilon_{ijk}\,\epsilon_{kmn}\,\epsilon_{npq} \iint_{\underset{\sim}{k}\omega} \frac{k_m \overline{B}_q\, \Phi_{jp}(\underset{\sim}{k},\omega)}{i\omega + \eta k^2}\, d\underset{\sim}{k}\, d\omega \qquad (3.3)$$

where the effects of initial conditions have been neglected. Also, assuming that the fluid is incompressible,

$$k_i \Phi_{ij}(k,\omega) = 0 = k_j \Phi_{ij}(k,\omega) \qquad (3.4)$$

(Batchelor, 1953, p. 27). Substituting (3.4) into (3.3), and making use of (2.82a),

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}\}_i = i\,\epsilon_{ij\ell} \iint_{\underset{\sim}{k}\omega} \frac{k_q \overline{B}_q\, \Phi_{j\ell}(\underset{\sim}{k},\omega)}{i\omega + \eta k^2}\, d\underset{\sim}{k}\, d\omega \qquad (3.5)$$

Clearly, if $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}$ is not to vanish, we must require

$$\epsilon_{ij\ell}\, \Phi_{j\ell} \not\equiv 0 \qquad (3.6)$$

But, from (3.2) and the nature of the spectrum tensor, (3.6) is equivalent to (3.1) - i.e. in order for a turbulent velocity field $\underset{\sim}{u}'$ to be able to maintain a magnetic field $\overline{\underset{\sim}{B}}$ which is nearly uniform and time-independent on the length and time scales of the turbulence, $\underset{\sim}{u}'$ *must have helicity*. (This derivation, of course, is only valid in the first order smoothing approximation.)

3.2 PT-invariant turbulence and nonuniform, oscillatory mean fields

3.2.1 The spectrum tensor for stationary, homogeneous, isotropic turbulence

It has recently been suggested by *Lerche (1971a,b,d, 1972b,d)* and *Lerche and Low (1971)* that helicity is not required when $\bar{\underset{\sim}{B}}$ is nonuniform and time-dependent on the length and time scales of the turbulence. These authors consider the case of stationary, homogeneous, *isotropic* turbulence, for which the spectrum tensor Φ_{ij} has the simple form *(Batchelor, 1953, p. 49)*

$$\Phi_{ij}(\underset{\sim}{k},\omega) = \frac{E(k,\omega)}{4\pi k^4}\left\{k^2\delta_{ij} - k_i k_j\right\} \tag{3.7}$$

In (3.7), which is valid for *incompressible flow*, $E(k,\omega)$ denotes the *energy spectrum function* of the turbulence. This function represents the density of contributions to the kinetic energy of the fluid in (k,ω) space, and

$$\int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} dk \; E(k,\omega) = \tfrac{1}{2}\,\overline{u_i'(\underset{\sim}{x},t)\,u_i'(\underset{\sim}{x},t)} \tag{3.8}$$

is the total kinetic energy per unit mass of the fluid. Consequently, *$E(k,\omega)$ can never be negative.*

It can be shown *(Batchelor, 1953, pp. 39-40, 51)* that $E(k,\omega)$ has the form

$$E(k,\omega) = C(\omega)\,k^4 + \mathcal{O}(k^6) \tag{3.9}$$

near $k = 0$.

3.2.2 PT-invariance and helicity

Since the spectrum tensor (3.7) is symmetric, it cannot satisfy (3.6) - i.e. *stationary, homogeneous, isotropic turbulence has no helicity.* This lack of helicity is a general property of PT-invariant turbulence *(see section 2.2).* As was shown in equations (2.4'), (2.8), and (2.8'), the correlation tensor for PT-invariant turbulence satisfies

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ji}(-\underset{\sim}{r},-\tau) \tag{3.10}$$

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ij}(-\underset{\sim}{r},-\tau) \tag{3.11}$$

$$R_{ij}(\underset{\sim}{r},\tau) = R_{ji}(\underset{\sim}{r},\tau) \tag{3.12}$$

It follows immediately from (3.12) and the definition of the spectrum tensor Φ_{ij} that

$$\Phi_{ij}(\underset{\sim}{k},\omega) = \Phi_{ji}(\underset{\sim}{k},\omega) \tag{3.13}$$

Since Φ_{ij} must also satisfy the condition of *Hermitian symmetry*

$$\Phi_{ij}(\underset{\sim}{k},\omega) = \Phi^{*}_{ji}(\underset{\sim}{k},\omega) \tag{3.13'}$$

as a consequence of (3.10), *the spectrum tensor for PT-invariant turbulence is purely real.* The converse is also true: *turbulence for which the spectrum tensor is purely real is necessarily PT-invariant.*

3.2.3 An anti-dynamo theorem for stationary, homogeneous, isotropic turbulence in an incompressible fluid

PT-invariant turbulence of the type (3.11) cannot maintain a nearly uniform $\bar{\underset{\sim}{B}}$. However, it is still possible that the turbulence might maintain a field of the form

$$\bar{\underset{\sim}{B}}(\underset{\sim}{x},t) = \hat{\bar{\underset{\sim}{B}}}(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x}+\Omega t\}} \tag{3.14}$$

for a suitable choice of $(\underset{\sim}{K},\Omega)$. The necessary condition for the maintenance of this field is that $(\underset{\sim}{K},\Omega)$ satisfy the dispersion relation (2.97)-(2.98) with $\mathrm{Im}\,\Omega \leqslant 0$. Rewriting the dispersion relation for the case of stationary, homogeneous, isotropic turbulence, making use of (2.88), (3.4), and (3.10), we have

$$\mathcal{I}m\,\Omega - \eta K^2 = \mathcal{R}e\, I^{(1)}(\underset{\sim}{K},\Omega) \tag{3.15}$$

$$\mathcal{R}e\,\Omega = -\mathcal{I}m\, I^{(1)}(\underset{\sim}{K},\Omega) \tag{3.16}$$

where

$$I^{(1)}(\underset{\sim}{K},\Omega) \equiv K_j K_p \iint_{\underset{\sim}{k}\,\omega} \frac{E(k,\omega)}{4\pi k^4} \frac{\{k^2\delta_{jp} - k_j k_p\}}{\{i(\omega+\Omega) + \eta(\underset{\sim}{k}+\underset{\sim}{K})^2\}}\, d\underset{\sim}{k}\, d\omega \tag{3.17}$$

Taking $\underset{\sim}{K}$ to define the z-axis in k-space, so that $\underset{\sim}{k}\cdot\underset{\sim}{K} = kK\cos\theta$,

$$K_j K_p\{k^2\delta_{jp} - k_j k_p\} = (kK)^2 - (\underset{\sim}{k}\cdot\underset{\sim}{K})^2 = (kK)^2\sin^2\theta \tag{3.18}$$

and (3.17) becomes

$$I^{(1)}(\underset{\sim}{K},\Omega)$$

$$= \tfrac{1}{2}K^2\int_{-\infty}^{+\infty} d\omega \int_0^{\infty} E(k,\omega)\,dk \int_0^{\pi} \frac{\sin^3\theta\, d\theta}{i(\omega+\Omega)+\eta(k^2+K^2+2kK\cos\theta)}$$

$$= \tfrac{1}{4}K^7\int_{-\infty}^{\infty} d\left(\frac{\omega}{\eta K^2}\right)\int_0^{\infty} d\left(\frac{k}{K}\right)\frac{E(k,\omega)}{k^4}\,\Theta\left\{\frac{k}{K},\frac{\omega+\mathrm{Re}\,\Omega}{\eta K^2}\,;\,\mathrm{Im}\,\frac{\Omega}{\eta K^2}\right\} \tag{3.19}$$

where

$$\Theta(\xi,\nu;p) \equiv 2\xi^4\int_0^{\pi}\frac{\sin^3\theta\,d\theta}{(1+\xi^2+2\xi\cos\theta-p)+i\nu} \tag{3.20}$$

It is clear from (3.15) that $\mathrm{Im}\,\Omega \leqslant 0$ if and only if $\mathrm{Re}\,I^{(1)} \leqslant -\eta K^2$. But, from (3.19), $\mathrm{Re}\,I^{(1)}$ can only be negative if $\mathrm{Re}\,\Theta < 0$ for at least <u>some</u> values of its arguments, since $E(k,\omega)$ is everywhere positive. From (3.20) we see that

$$\mathrm{Re}\,\Theta(\xi,\nu;p) = 2\xi^4\int_0^{\pi}\frac{(1+\xi^2+2\xi\cos\theta-p)}{(1+\xi^2+2\xi\cos\theta-p)^2+\nu^2}\,\sin^3\theta\,d\theta$$

$$> 2\xi^4\int_0^{\pi}\frac{\{(1-\xi)^2-p\}\,\sin^3\theta}{(1+\xi^2+2\xi\cos\theta-p)^2+\nu^2}\,d\theta \tag{3.21}$$

Thus, when $p < 0$, corresponding to $\mathrm{Im}\,\Omega < 0$ in (3.19), $\mathrm{Re}\,\Theta(\xi,\nu;p)$ is a non-negative function of ξ and ν . It follows that $\mathrm{Re}\,I^{(1)}$ can *never* be negative when $\mathrm{Im}\,\Omega < 0$. We have therefore proved that *stationary, homogeneous, isotropic turbulence in an incompressible fluid cannot support a growing magnetic field of the type (3.14), within the framework of the first order smoothing approximation.* An alternative proof of this statement has been

given by *Krause and Roberts (1973).*

3.2.4 Extension of the anti-dynamo theorem - PT-invariant turbulence in an incompressible fluid

We may extend the proof given above to show that in an incompressible fluid, a growing magnetic field of the form (3.14) cannot be supported by *any* stationary, homogeneous turbulence which is PT-invariant. Turbulence of this sort will have a spectrum tensor which satisfies both (3.13) and the condition of Hermitian symmetry

$$\Phi_{ij}(\underset{\sim}{k},\omega) \quad = \quad \Phi^{*}_{ji}(\underset{\sim}{k},\omega) \tag{3.22}$$

As noted above, the spectrum tensor must therefore be purely real. From (3.15) we see again that $\operatorname{Im}\Omega \leqslant 0$ if and only if $\operatorname{Re} I^{(1)}$, defined by (2.88), is less than or equal to $-\eta K^2$. But, from (2.88), (3.4), and the condition that $\Phi_{jp}(\underset{\sim}{k},\omega)$ is real,

$$\operatorname{Re} I^{(1)}(\underset{\sim}{K},\Omega) = K_j K_p \iint\limits_{\underset{\sim}{k}\,\omega} \frac{\Phi_{jp}(\underset{\sim}{k},\omega)\{\eta(\underset{\sim}{k}+\underset{\sim}{K})^2 - \operatorname{Im}\Omega\}}{(\omega + \operatorname{Re}\Omega)^2 + \{\eta(\underset{\sim}{k}+\underset{\sim}{K})^2 - \operatorname{Im}\Omega\}^2}\, d\underset{\sim}{k}\, d\omega \tag{3.23}$$

This integral can be negative when $\operatorname{Im}\Omega \leqslant 0$ only if

$$K_j K_p \;\Phi_{jp}(\underset{\sim}{k},\omega) \quad < \quad 0 \tag{3.24}$$

for some range of $(\underset{\sim}{k},\omega)$. However, by *Bochner's Theorem (see Krause and Roberts, 1973; Batchelor, 1953, p. 25),*

the quadratic form

$$\Phi \equiv X_j X_p^* \Phi_{jp}(\underset{\sim}{k},\omega) \tag{3.25}$$

must be non-negative for <u>any</u> choice of the complex vector $\underset{\sim}{X}$, if Φ_{jp} is the spectrum tensor of a continuous, stationary, homogeneous random process. Thus (3.24) cannot be satisfied for <u>any</u> choice of $(\underset{\sim}{k},\omega)$ and the proof is complete.

3.2.5 <u>Failure of the anti-dynamo theorem for PT-invariant in a compressible fluid</u>

The proof cannot be extended to the case of stationary, homogeneous, PT-invariant turbulence in a *compressible* fluid. To see this, we may note that, by virtue of (3.13) and (3.22), the spectrum tensor is still purely real in the compressible case. Thus (2.88) gives

$$\mathcal{R}e\, I^{(1)}(\underset{\sim}{K},\Omega) = \iint\limits_{\underset{\sim}{k}\omega} \frac{K_j(K_p+k_p)\Phi_{jp}(\underset{\sim}{k},\omega)\{\eta(\underset{\sim}{k}+\underset{\sim}{K})^2 - \mathcal{I}m\,\Omega\}}{(\omega+\mathcal{R}e\,\Omega)^2+\{\eta(\underset{\sim}{k}+\underset{\sim}{K})^2-\mathcal{I}m\,\Omega\}^2}\, d\underset{\sim}{k}\, d\omega \tag{3.26}$$

and $\mathrm{Re}\ I^{(1)}$ can be negative for $\mathrm{Im}\ \Omega \leqslant 0$ only if

$$K_j(K_p + k_p)\ \Phi_{jp}(\underset{\sim}{k},\omega) < 0 \tag{3.27}$$

for some range of $(\underset{\sim}{k},\omega)$. From (3.13),

$$k_j K_p\ \Phi_{jp} = k_p K_j\ \Phi_{jp} \tag{3.28}$$

for PT-invariant turbulence. Therefore,

$$K_j(K_p+k_p)\,\Phi_{jp} = \frac{1}{2}\left\{(K_j+k_j)(K_p+k_p) + K_jK_p - k_jk_p\right\}\Phi_{jp} \qquad (3.29)$$

By Bochner's Theorem (3.25), each of the three quadratic forms on the right hand side of (3.29) is non-negative. However, the minus sign attached to the last term is sufficient to allow the right hand side to become negative. Consider, for example, the case $\Phi_{jp} = \Phi^\circ\delta_{jp}$, where $\Phi^\circ > 0$. In this case, (3.29) becomes

$$K_j(K_p+k_p)\,\Phi_{jp} \equiv K_j(K_p+k_p)\,\Phi^\circ\delta_{jp} = \Phi^\circ \underset{\sim}{K}\cdot\{\underset{\sim}{K}+\underset{\sim}{k}\} \qquad (3.30)$$

It is clear from (3.30) that if $\underset{\sim}{k}\cdot\underset{\sim}{K} < 0$ and $|\underset{\sim}{k}\cdot\underset{\sim}{K}| > |\underset{\sim}{K}|^2$ equation (3.27) *will* be satisfied. It follows that it may be possible for stationary, homogeneous, PT-invariant turbulence in a compressible fluid to support a magnetic field of the form (3.14), within the framework of the first order smoothing approximation. However, *Krause and Roberts (1973)* show that stationary, homogeneous, isotropic turbulence in a compressible fluid cannot support dynamo action in the case when

$$q \equiv \lambda_c^2/\eta\tau_c \gg 1 \qquad (3.31)$$

where λ_c and τ_c are the correlation length and time of the turbulence.

3.2.6 Reconciliation of the anti-dynamo theorem with the work of *Lerche and Low*

The theorem proved in *sections 3.2.3 and 3.2.4* is in direct contradiction to the work of *Lerche and Low (1971)*, who suggest that stationary homogeneous, isotropic turbulence in an incompressible fluid can lead to dynamo action for a mean field $\bar{\underset{\sim}{B}}$ of the form (3.14), within the framework of first order smoothing. The discrepancy between the two results lies in the fact that the *correlation tensor* used by *Lerche and Low* does not satisfy *Bochner's Theorem*. In fact, their correlation tensor corresponds to a spectrum tensor of the form (3.7) in which the energy spectrum function $E(k,\omega)$ has *negative* values for some choices of $(\underset{\sim}{k},\omega)$. Such a choice of $E(k,\omega)$ is clearly unphysical.

Krause (1972a) and *Krause and Roberts (1973)* give a more detailed discussion of the work of *Lerche (1971a-f, 1972a-d)* and *Lerche and Low (1971)*, in which attention is drawn to the discrepancy mentioned here and to several other inconsistencies.

3.2.7 The possibility of dynamo action with "mirror-symmetric" turbulence

It has frequently been stated in the literature that "mirror-symmetric" turbulence cannot support dynamo action *(e.g. Moffatt, 1970a)*. This statement is usually supported

by considering the case of isotropic turbulence (3.7). However, as noted in *Table 11, isotropy implies PT-invariance*, and, as proved in the last few sections, it is the property of PT-invariance that leads to the impossibility of dynamo action (at least under certain conditions). The anti-dynamo theorem proved in *sections 3.2.3 and 3.2.4* cannot be extended to the case of P-invariance (or "mirror symmetry"), since there is no general requirement that the spectrum tensor of P-invariant turbulence be symmetric under interchange of indices. It is therefore possible that some types of "mirror-symmetric" turbulence in an incompressible fluid can produce dynamo action, even within the framework of first order smoothing. The only types of P-invariant turbulence which might be able to do so are those whose average properties are not invariant under time reversal - *i.e. the turbulence must not be T-invariant.* Furthermore, as noted above, *the turbulence must not be isotropic.*

3.3 PT-invariant turbulence and decaying mean fields

3.3.1 The effect of initial conditions

Gilliland and Aldridge (1973) have discussed the case of PT-invariant turbulence in more detail, attention being given to mean fields of the type (3.14) which *decay* with time (i.e. $\operatorname{Im}\Omega > 0$). Under these circumstances it is no longer possible to neglect the effect of initial conditions on $\underset{\sim}{B}'$, and the term on the right hand side of (2.80) must be retained. When the turbulence is PT-invariant, so that (3.13) holds, (2.80) reduces to

$$\{i\Omega + \eta K^2 + I^{(1)}(\underset{\sim}{K},\Omega)\}\, \hat{\bar{B}}_r(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x}+\Omega t\}}$$

$$= -\epsilon_{rsi}\epsilon_{ij\ell} \iiint\limits_{\underset{\sim}{k}\,\omega\,\underset{\sim}{k}'} (\underset{\sim}{k}'-\underset{\sim}{k})_s\, \overline{dZ_j^*(\underset{\sim}{k},\omega)\, dY'_{0\ell}(\underset{\sim}{k}')} \cdot e^{i(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} - (\eta k'^2 - i\omega)t} \qquad (3.32)$$

The term on the right hand side of (3.32) is not in a useful form. An alternative expression may be obtained by starting from the Green's function representation (2.22), with $\underset{\sim}{\bar{u}} = 0$ and G_{ij} defined by (2.61). Substituting the Fourier representations (2.63) and (3.14) for R_{ij} and $\underset{\sim}{\bar{B}}$ in this equation, and making use of (3.13) we obtain

$$\left\{i\Omega + \eta K^2 + I^{(1)}(\underset{\sim}{K},\Omega)\right\}\hat{\bar{B}}_r(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x}+\Omega t\}}$$

$$= -\epsilon_{rsi}\epsilon_{ij\ell}\left\{K_s \hat{\bar{B}}_\ell\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x}+\Omega t\}}\,\cdot\right.$$

$$\cdot \iint\limits_{\underset{\sim}{k}\,\omega} \frac{(\underset{\sim}{k}+\underset{\sim}{K})_p\, \Phi_{jp}(\underset{\sim}{k},\omega)}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\, e^{-[i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2](t-t_o)}\, d\underset{\sim}{k}\, d\omega$$

$$- \int\limits_{\text{all space}} d\underset{\sim}{r}\; e^{-r^2/4\eta(t-t_o)}\,[4\pi\eta(t-t_o)]^{-3/2}\,\cdot$$

$$\left.\cdot\ \frac{\partial}{\partial x_s}\left[u'_j(\underset{\sim}{x},t)\, B'_{o_\ell}(\underset{\sim}{x}-\underset{\sim}{r},t_o)\right]\right\} \qquad (3.33)$$

3.3.2 Initial condition *I* - $\underset{\sim}{B}'$ independent of $\bar{\underset{\sim}{B}}$

We may now consider the effect of various initial conditions on the form of the dispersion relation. If

$$\overline{u'_j(\underset{\sim}{x},t)\, B'_{o_\ell}(\underset{\sim}{x}',t_o)} = 0 \qquad (3.34)$$

or, less restrictively, if

$$\overline{u'_j(\underset{\sim}{x},t)\, B'_{o_\ell}(\underset{\sim}{x}',t_o)} = f_{j\ell}(\underset{\sim}{x}'-\underset{\sim}{x}\,;t,t_o) \qquad (3.34')$$

equation (3.33) reduces to

$$i\Omega + \eta K^2 = -K_j \iint\limits_{\underset{\sim}{k}\,\omega} \frac{(\underset{\sim}{k}+\underset{\sim}{K})_p\, \Phi_{jp}(\underset{\sim}{k},\omega)}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\,\cdot$$

$$\cdot\left\{1 - e^{[i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2](t-t_o)}\right\} d\underset{\sim}{k}\, d\omega \qquad (3.35)$$

or

$$i\Omega + \eta K^2$$

$$= -K_j \int_0^{t-t_0} e^{-i\Omega\tau} d\tau \int_{\underset{\sim}{k}} (\underset{\sim}{k}+\underset{\sim}{K})_p \hat{R}_{jp}(\underset{\sim}{k};\tau) e^{-\eta(\underset{\sim}{k}+\underset{\sim}{K})^2\tau} d\underset{\sim}{k} \qquad (3.35')$$

where $R_{ij}(\underset{\sim}{k};\tau)$ is the spatial Fourier transform of the correlation tensor R_{ij}, defined by

$$R_{ij}(\underset{\sim}{r},\tau) = \int_{\underset{\sim}{k}} \hat{R}_{ij}(\underset{\sim}{k};\tau) e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} d\underset{\sim}{k} \qquad (3.36)$$

The "dispersion relation" (3.35) or (3.35') is clearly *time-dependent*. If $\mathrm{Im}\,\Omega > 0$, values of $\underset{\sim}{k}$ such that

$$\eta(\underset{\sim}{k} + \underset{\sim}{K})^2 < \mathrm{Im}\,\Omega \qquad (3.37)$$

will provide contributions to the integral on the right hand side of (3.35') which do not tend to zero as $(t-t_o)$ goes to infinity.

3.3.3 <u>Initial condition *II* - $\underset{\sim}{B}'$ correlated with $\overline{\underset{\sim}{B}}$</u>

A second possible initial condition is that the right hand side of (3.32) or (3.33) should vanish. This condition is satisfied if

$$\overline{dZ_j^*(\underset{\sim}{k},\omega)\, dY'_{o_\ell}(\underset{\sim}{k}')} = \zeta_{j\ell}(\underset{\sim}{k},\omega)\, \delta(\underset{\sim}{k}-\underset{\sim}{k}')\, d\underset{\sim}{k}\, d\underset{\sim}{k}'\, d\omega \qquad (3.38)$$

or

$$\{\mathrm{curl}\, (\overline{\underset{\sim}{u}'\times \underset{\sim}{B}'_o})\}_r(\underset{\sim}{x},t_o) = -\hat{\overline{B}}_r(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x}+\Omega t_o\}}\, I^{(1)}(\underset{\sim}{K},\Omega) \qquad (3.38')$$

or

$$\overline{u_j'(\underset{\sim}{x},t)\, B_{o\ell}'(\underset{\sim}{x}',t_o)} = -i\,\hat{\overline{B}}_\ell(\underset{\sim}{K},\Omega)\, e^{i\{\underset{\sim}{K}\cdot\underset{\sim}{x}+\Omega t\}} \cdot$$

$$\cdot \iint_{\underset{\sim}{k}\omega} \frac{(\underset{\sim}{k}+\underset{\sim}{K})_p\, \Phi_{jp}(\underset{\sim}{k},\omega)}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2} \cdot \tag{3.38"}$$

$$\cdot\, e^{-i\{(\underset{\sim}{k}+\underset{\sim}{K})\cdot(\underset{\sim}{x}-\underset{\sim}{x}')+(\omega+\Omega)(t-t_o)\}}\, d\underset{\sim}{k}\, d\omega$$

implying that the fluctuating fields $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ are correlated even at time $t = t_o$, and that $\underset{\sim}{B}'$ is always proportional to $|\overline{\underset{\sim}{B}}|$. Under these circumstances, the dispersion relation (3.15)-(3.16) is valid for <u>all</u> Ω .

3.4 PT-invariant turbulence and decaying mean fields - initial condition *II*

3.4.1 The mean field dispersion relation

Let us first examine the case when *initial condition II* (3.38) applies. The dispersion relation is, rewriting (3.15) and (3.16),

$$i\Omega + \eta K^2 + I^{(1)}(\underset{\sim}{K},\Omega) = 0 \tag{3.39}$$

where

$$I^{(1)}(\underset{\sim}{K},\Omega) \equiv \iint\limits_{\underset{\sim}{k}\,\omega} \frac{K_j\,\Phi_{jp}(\underset{\sim}{k},\omega)(\underset{\sim}{k}+\underset{\sim}{K})_p}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\,d\underset{\sim}{k}\,d\omega \tag{3.40}$$

When $\mathrm{Im}\,\Omega \geqslant 0$, there is a pole on the integration contour in (3.40), at

$$\omega = -\Omega + i\eta(\underset{\sim}{k}+\underset{\sim}{K})^2 \tag{3.41}$$

and the integral must be defined in terms of a *Cauchy principal value*.

3.4.2 The mean field dispersion relation for isotropic turbulence in an incompressible fluid

When the turbulence is isotropic and the fluid is incompressible, the spectrum tensor is given by (3.7) and the integral (3.40) may be written in the form (3.19). We may then discuss the behaviour of the integral in terms of

the function $\Theta(\xi,\nu;p)$ defined in (3.20). For all $p < 1$ $\mathrm{Re}\,\Theta$ is a non-negative function of ξ and ν, going to zero along the line $\xi = 0$. In addition, $\mathrm{Re}\,\Theta$ is an even function of ν, tending monotonically to zero as $|\nu| \to \infty$. The behaviour of $\mathrm{Re}\,\Theta$ with ξ is indicated in *Figure 1*, where the function is plotted against ξ for $\nu = 0$, for several typical values of p. From this behaviour it is clear that the dispersion relation (3.39) has no solutions for $\mathrm{Im}\,\Omega < \eta K^2$, since $\mathrm{Re}\,I^{(1)}$ is positive definite in this region, while the real part of (3.39) requires that

$$\mathrm{Im}\,\Omega - \eta K^2 = \mathrm{Re}\,I^{(1)}(\underset{\sim}{K},\Omega) \tag{3.15}$$

Solutions will occur, however, when $\mathrm{Im}\,\Omega > \eta K^2$, since now the left hand side of (3.15) is positive, while the right hand side can assume positive or negative values, according to the choice of (K,Ω).

The real and imaginary parts of Θ are

$$\mathrm{Re}\,\Theta(\xi,\nu;p) = 2\xi^4\int_0^{\pi} \frac{(1+\xi^2+2\xi\cos\theta-p)\sin^3\theta}{(1+\xi^2+2\xi\cos\theta-p)^2+\nu^2}\,d\theta \tag{3.42}$$

$$\mathrm{Im}\,\Theta(\xi,\nu;p) = -2\nu\xi^4\int_0^{\pi} \frac{\sin^3\theta}{(1+\xi^2+2\xi\cos\theta-p)^2+\nu^2}\,d\theta \tag{3.43}$$

These functions are plotted against ξ and ν for typical values of $p > 1$ in *Figure 2*.

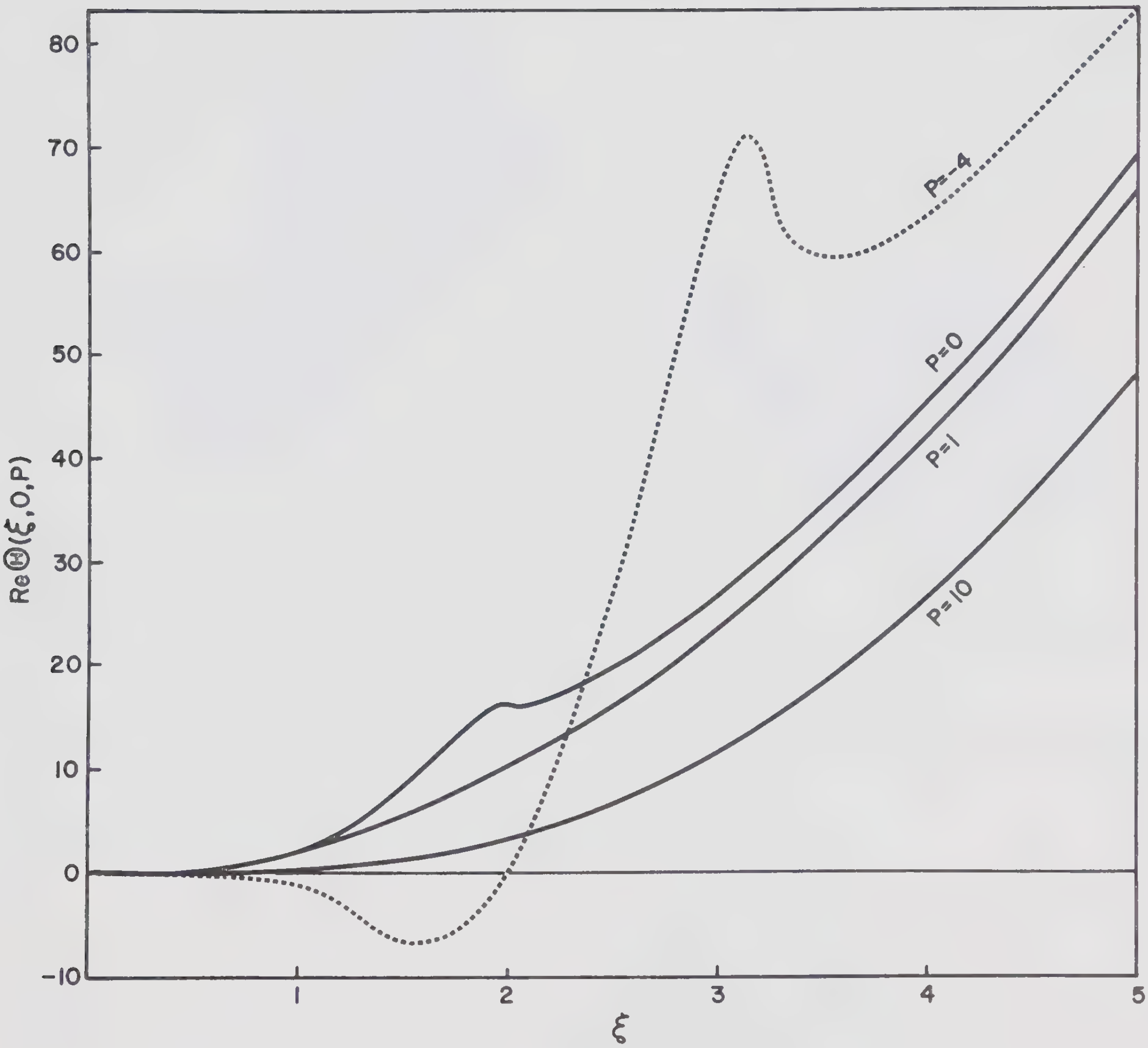
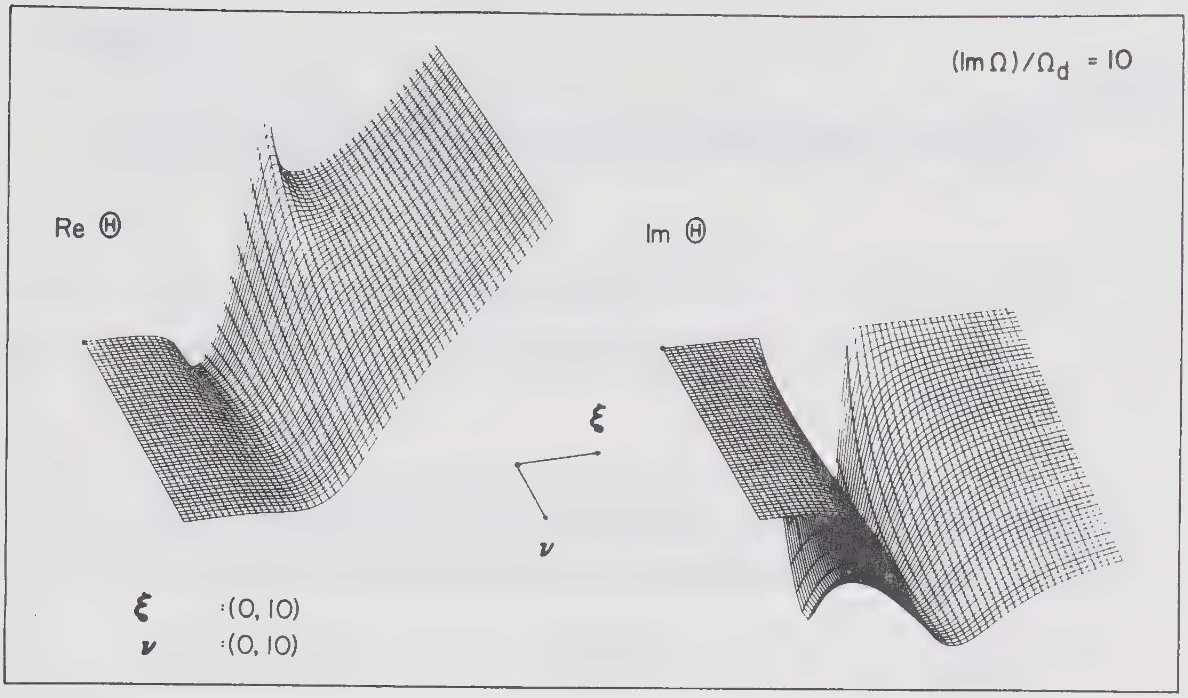


Figure 1. Re $\Theta(\xi,0;p)$ as a function of ξ for various p .

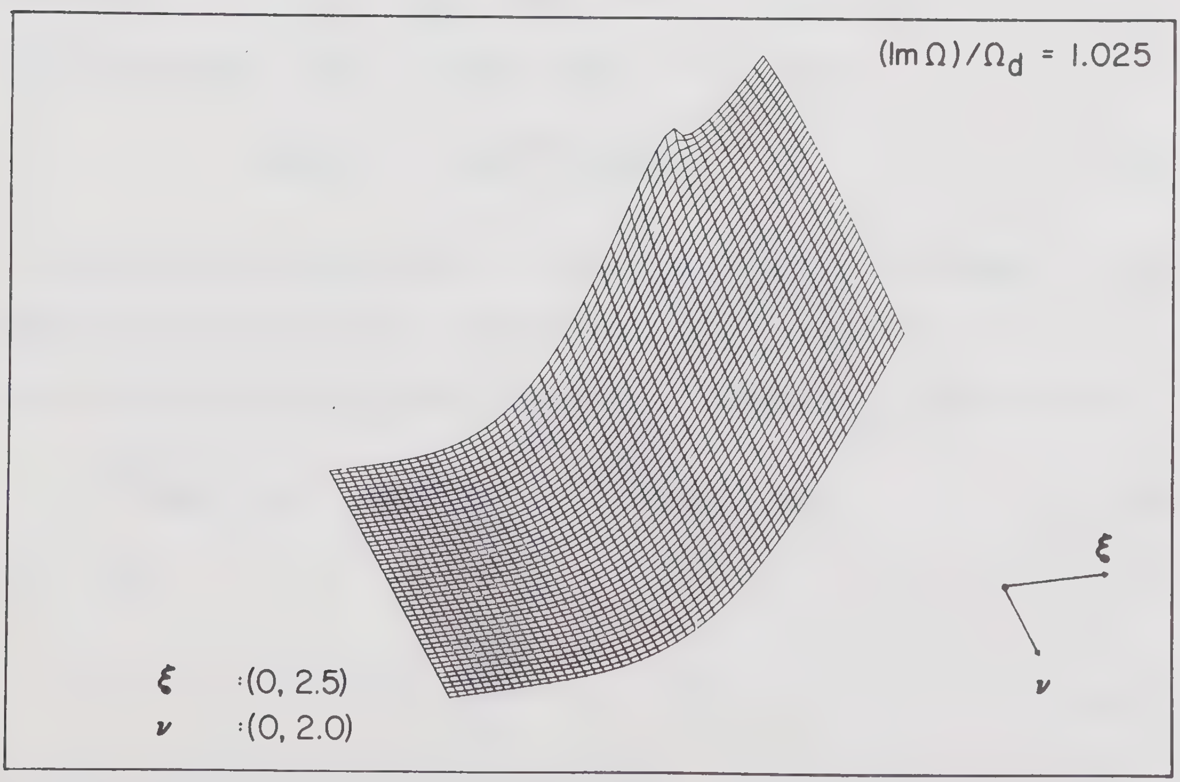
Figure 2. $\Theta(\xi,\nu;p)$ as a function of ξ and ν for two values of p.

In the plots shown, p has been replaced by $(\mathrm{Im}\ \Omega)/\Omega_d$, where $\Omega_d \equiv \eta K^2$. In the upper plot, $\mathrm{Re}\ \Theta$ and $\mathrm{Im}\ \Theta$ are plotted against ξ and ν for $p = 10$, while in the lower plot, $\mathrm{Re}\ \Theta$ is plotted against ξ and ν for $p = 1.025$.

$\Theta(\xi, \nu; \mathrm{Im}\Omega/\Omega_d)$



$\mathrm{Re}\ \Theta(\xi, \nu; \mathrm{Im}\Omega/\Omega_d)$



From (3.19),

$$I^{(1)}(\underset{\sim}{K},\Omega) = \frac{1}{4}K^7\int_{-\infty}^{\infty}d\nu\int_0^{\infty}d\xi\,\frac{1}{(K\xi)^4}\,E(K\xi,\eta K^2\nu+\mathcal{R}e\,\Omega)\,\Theta(\xi,\nu;\mathcal{I}m\,\frac{\Omega}{\eta K^2}) \tag{3.44}$$

Since $E(k,\omega)$ is purely real, Re $I^{(1)}$ is determined by Re Θ, while Im $I^{(1)}$ is determined by Im Θ.

3.4.3 The nature of the dispersion relation - contrast between oscillatory and non-oscillatory mean fields

Since Im Θ is an odd function of ν, while $E(k,\omega)$ is by definition an even function of ω, it follows from (3.44) that Im $I^{(1)} \equiv 0$ whenever Re $\Omega = 0$. In this case, the imaginary part of (3.39),

$$\mathcal{R}e\,\Omega = -\,\mathcal{I}m\,I^{(1)}(\underset{\sim}{K},\Omega) \tag{3.16}$$

is satisfied identically. When Re $\Omega \neq 0$, however, (3.16) is no longer trivially satisfied, and (3.15) and (3.16) may be considered as an eigenvalue problem. If we define

$$A\cdot\hat{h}(k,\omega) \equiv E(k,\omega)/k^4 \quad , \quad \hat{h}(0,0) \equiv 1 \tag{3.45}$$

and let

$$P \equiv 1 + \frac{i\Omega}{\eta K^2} \tag{3.46}$$

(3.39) can be written in the form

$$\mathcal{A}\begin{pmatrix}\mathrm{Re}\,P\\ \mathrm{Im}\,P\end{pmatrix} = \frac{\eta}{AK^5}\begin{pmatrix}\mathrm{Re}\,P\\ \mathrm{Im}\,P\end{pmatrix} \tag{3.47}$$

where $\mathcal{A}$ is a complex-valued integral operator. The parameter η/AK^5 may then be considered as an eigenvalue for the problem (3.47).

Solutions to the dispersion relation (3.39) are thus of two types - those with $\mathrm{Re}\,\Omega = 0$, and those with $\mathrm{Re}\,\Omega \neq 0$. In the limit as $\mathrm{Re}\,\Omega \to 0$, solutions of the second type may form a discrete subset of the set of solutions of the first type.

3.5 Digression: the properties of stationary, homogeneous, isotropic turbulence

3.5.1 General properties

Before continuing with a more detailed study of the solutions of the mean field dispersion relation, let us consider the significance of the function (3.45). For *isotropic* turbulence, the correlation tensor is of the form

$$R_{ij}(\underset{\sim}{r},\tau) = F(r,\tau)\, r_i r_j + G(r,\tau)\, \delta_{ij} \tag{3.48}$$

(Batchelor, 1953, p. 45ff.), and we may define *longitudinal and transverse correlation functions* $f(r,\tau)$ and $g(r,\tau)$ such that

$$\overline{u_{(p)}^2}\, f(r,\tau) \equiv \overline{u_{(p)}(\underset{\sim}{x},t)\, u_{(p)}(\underset{\sim}{x}+\underset{\sim}{r},t+\tau)} = r^2 F(r,\tau) + G(r,\tau) \tag{3.49}$$

$$\overline{u_{(n)}^2}\, g(r,\tau) \equiv \overline{u_{(n)}(\underset{\sim}{x},t)\, u_{(n)}(\underset{\sim}{x}+\underset{\sim}{r},t+\tau)} = G(r,\tau) \tag{3.50}$$

$u_{(p)}$ and $u_{(n)}$ denote velocity components parallel and normal respectively to the vector separation $\underset{\sim}{r}$ *(Batchelor, 1953, p. 46)*, and

$$\overline{u_{(p)}^2} \equiv \overline{u_{(n)}^2} \equiv \tfrac{1}{3}\, \overline{u_i'(\underset{\sim}{x},t)\, u_i'(\underset{\sim}{x},t)} \equiv \tfrac{1}{3}\, \overline{u^2} \tag{3.51}$$

The Fourier transforms of R_{ij}, f, and g are

$$\hat{R}_{ij}(\underset{\sim}{k},\omega) \equiv \Phi_{ij}(\underset{\sim}{k},\omega) = -\frac{\partial^2 \hat{F}(k,\omega)}{\partial k_i\, \partial k_j} + \hat{G}(k,\omega)\, \delta_{ij}$$

$$= -\frac{1}{k}\frac{\partial}{\partial k}\left\{\frac{1}{k}\frac{\partial \hat{F}}{\partial k}\right\} k_i k_j + \left\{\hat{G} - \frac{1}{k}\frac{\partial \hat{F}}{\partial k}\right\}\delta_{ij} \tag{3.52}$$

$$\frac{1}{3}\overline{u^2}\,\hat{f}(k,\omega) = -\nabla_{\underset{\approx}{k}}^2 \hat{F}(k,\omega) + \hat{G}(k,\omega)$$

$$= -\frac{1}{k^2}\frac{\partial}{\partial k}\left\{k^2 \frac{\partial \hat{F}}{\partial k}\right\} + \hat{G} \qquad (3.53)$$

$$\frac{1}{3}\overline{u^2}\,\hat{g}(k,\omega) = \hat{G}(k,\omega) \qquad (3.54)$$

3.5.2 Isotropic turbulence in an incompressible fluid

For incompressible flow, the spectrum tensor Φ_{ij} must satisfy (3.4), so that

$$-k\frac{\partial}{\partial k}\left\{\frac{1}{k}\frac{\partial \hat{F}}{\partial k}\right\} + \left\{\hat{G} - \frac{1}{k}\frac{\partial \hat{F}}{\partial k}\right\} = -\frac{\partial^2 \hat{F}}{\partial k^2} + \hat{G} = 0$$

giving

$$\hat{G}(k,\omega) = \frac{\partial^2}{\partial k^2}\hat{F}(k,\omega) \qquad (3.55)$$

From (3.52) and (3.55),

$$\frac{1}{3}u^2\,\hat{f}(k,\omega) = -\frac{2}{k}\frac{\partial}{\partial k}\hat{F}(k,\omega)$$

so that, for incompressible flow,

$$\frac{1}{k}\frac{\partial \hat{F}}{\partial k} = -\frac{1}{6}\overline{u^2}\,\hat{f} \qquad (3.56)$$

$$\hat{G} = \frac{1}{3}\overline{u^2}\,\hat{g} = \frac{\partial^2 \hat{F}}{\partial k^2} = -\frac{1}{6}\overline{u^2}\frac{\partial}{\partial k}(k\hat{f}) \qquad (3.57)$$

$$\Phi_{ij} = \frac{1}{6}\overline{u^2}\frac{1}{k}\frac{\partial \hat{f}}{\partial k}k_i k_j + \left\{-\frac{1}{6}\overline{u^2}\frac{\partial}{\partial k}(k\hat{f}) + \frac{1}{6}\overline{u^2}\hat{f}\right\}\delta_{ij}$$

$$= -\frac{1}{6}\overline{u^2}\frac{1}{k}\frac{\partial \hat{f}}{\partial k}\left\{k^2\delta_{ij} - k_i k_j\right\} \qquad (3.58)$$

3.5.3 $E(k,\omega)/k^4$ for isotropic turbulence in an incompressible fluid

From (3.7), the energy spectrum function $E(k,\omega)$ is given by

$$E(k,\omega) = 2\pi k^2 \Phi_{ii}(\underset{\sim}{k},\omega) \tag{3.59}$$

Thus, by (3.58) and (3.59),

$$E(k,\omega) = -\frac{2\pi}{3}\overline{u^2}\, k^3 \frac{\partial}{\partial k}\hat{f}(k,\omega) \tag{3.60}$$

The function (3.45) can therefore be represented as

$$A\cdot\hat{h}(k,\omega) \equiv E(k,\omega)/k^4 = -\frac{2\pi}{3}\overline{u^2}\frac{1}{k}\frac{\partial}{\partial k}\hat{f}(k,\omega) \tag{3.61}$$

where

$$A \equiv \left\{E(k,\omega)/k^4\right\}_{k=0=\omega} = -\frac{2\pi}{3}\overline{u^2}\left\{\frac{1}{k}\frac{\partial}{\partial k}\hat{f}(k,\omega)\right\}_{k=0=\omega} \tag{3.62}$$

From the definition of the Fourier transform, $\hat{f}(k,\omega)$ is given by

$$\hat{f}(k,\omega) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\tau \int_{\text{all space}} d\underset{\sim}{r}\; f(r,\tau)\, e^{-i\{\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega\tau\}}$$

$$= \frac{2}{(2\pi)^3}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\int_0^{\infty} dr\;\; r^2 f(r,\tau)\,\frac{\sin kr}{kr} \tag{3.63}$$

Thus

$$E(k,\omega)/k^4 = -\frac{2}{3(2\pi)^3}\overline{u^2}\int_{-\infty}^{\infty} e^{-i\omega\tau}d\tau\;\cdot$$

$$\cdot\int_0^{\infty} r^4 f(r,\tau)\frac{1}{(kr)^2}\left\{\cos kr - \frac{\sin kr}{kr}\right\}dr \tag{3.64}$$

The oscillatory factor in the integrand of (3.64) may be expanded in a Taylor series about $kr = 0$.

$$\frac{1}{(kr)^2}\left\{\cos kr - \frac{\sin kr}{kr}\right\} = -\frac{1}{3}\left\{1 - \frac{(kr)^2}{10} + \cdots\right\} \tag{3.65}$$

Substituting (3.65) into (3.64), and taking the limit $k \to 0$, $\omega \to 0$, we have, from (3.62), that

$$A = \left\{E(k,\omega)/k^4\right\}_{k=0=\omega} = \frac{2}{9(2\pi)^2}\,\overline{u^2}\int_{-\infty}^{\infty}d\tau\int_0^{\infty}r^4 f(r,\tau)\,dr \tag{3.66}$$

3.5.4 Gaussian isotropic turbulence in an incompressible fluid

The expressions (3.64) and (3.66) may be used to evaluate E/k^4 , $\hat{h}$, and A for any desired type of isotropic turbulence. Consider first the case when the longitudinal correlation function is *Gaussian*, so that

$$f(r,\tau) = e^{-\frac{1}{2}\left\{r^2/\lambda_c^2 + \tau^2/\tau_c^2\right\}} \tag{3.67}$$

λ_c and τ_c are the *correlation length* and *correlation time* of the turbulence.

Substituting (3.67) into (3.64) and (3.66), we obtain

$$E(k,\omega)/k^4 = \frac{1}{6\pi}\,\overline{u^2}\,\lambda_c^5\,\tau_c\,e^{-\frac{1}{2}\left\{(\lambda_c k)^2 + (\tau_c\omega)^2\right\}} \tag{3.68}$$

$$A = \frac{1}{6\pi}\,\overline{u^2}\,\lambda_c^5\,\tau_c \tag{3.69}$$

(See Gradshteyn and Ryzhik, 1965, equations 3.952.4, 3.952.1, and 3.461.2 for the integral formulae used in obtaining 3.68 and 3.69.) From (3.68), (3.69), (3.45), and (3.63),

$$\hat{h}(k,\omega) = e^{-\frac{1}{2}\{(\lambda_c k)^2 + (\tau_c \omega)^2\}} = \frac{(2\pi)^2}{\lambda_c^3 \tau_c} \hat{f}(k,\omega) \qquad (3.70)$$

3.5.5 <u>Gaussian-exponential isotropic turbulence in an incompressible fluid</u>

As a second example, consider the case in which the longitudinal correlation function is Gaussian in space and exponential in time.

$$f(r,\tau) = e^{-r^2/2\lambda_c^2 - |\tau|/\tau_c} \qquad (3.71)$$

From (3.64), (3.66), and (3.71),

$$E(k,\omega)/k^4 = \frac{1}{6\pi}\sqrt{\frac{2}{\pi}}\,\overline{u^2}\,\lambda_c^5 \tau_c \; \frac{e^{-\frac{1}{2}(\lambda_c k)^2}}{1+(\tau_c\omega)^2} \qquad (3.72)$$

$$A = \frac{1}{6\pi}\sqrt{\frac{2}{\pi}}\,\overline{u^2}\,\lambda_c^5 \tau_c \qquad (3.73)$$

(See Gradshteyn and Ryzhik, 1965, equations 3.952.4, 3.952.1, 3.461.2, and 3.310.) From (3.72), (3.73), (3.45), and (3.63),

$$\hat{h}(k,\omega) = \frac{1}{1+(\tau_c\omega)^2}\, e^{-\frac{1}{2}(\lambda_c k)^2} = \frac{(2\pi)^2\sqrt{\pi/2}}{\lambda_c^3 \tau_c}\, \hat{f}(k,\omega) \qquad (3.74)$$

The correlation functions (3.67) and (3.71) have been used fairly widely in the literature. *(See, for example, P.H. Roberts, 1971a; Krause and Rädler, 1971.)* The results obtained in (3.68)-(3.70) and (3.72)-(3.74) are summarized in *Table 12, p.* 149, along with similar results for "exponential-Gaussian" and "exponential" isotropic turbulence.

TABLE 12 - VARIOUS TYPES OF STATIONARY, HOMOGENEOUS, ISOTROPIC TURBULENCE

Turbulence	$f(r,\tau)$	$\hat{h}(k,\omega)$	C	$\hat{f}(k,\omega)/\hat{f}_o$	$\hat{f}_o/\lambda_c^3\tau_c$
Gaussian	$e^{-\frac{1}{2}\{r^2/\lambda_c^2 + \tau^2/\tau_c^2\}}$	$e^{-\frac{1}{2}\{(k\lambda_c)^2+(\omega\tau_c)^2\}}$	$\frac{1}{6\pi}$	$\hat{h}(k,\omega)$	$\frac{1}{(2\pi)^2}$
Gaussian-exponential	$e^{-\{r^2/2\lambda_c^2 + \lvert\tau\rvert/\tau_c\}}$	$\frac{e^{-\frac{1}{2}(k\lambda_c)^2}}{1+(\omega\tau_c)^2}$	$\frac{1}{6\pi}\sqrt{\frac{2}{\pi}}$	$\hat{h}(k,\omega)$	$\frac{1}{(2\pi)^2}\sqrt{\frac{2}{\pi}}$
Exponential-Gaussian	$e^{-\{r/\lambda_c + \tau^2/2\tau_c^2\}}$	$\frac{e^{-\frac{1}{2}(\omega\tau_c)^2}}{\{1+(k\lambda_c)^2\}^3}$	$\frac{4}{3\pi}\sqrt{\frac{1}{2\pi}}$	$\{1+(k\lambda_c)^2\}\hat{h}(k,\omega)$	$\frac{1}{(2\pi)^2}\sqrt{\frac{2}{\pi}}$
Exponential	$e^{-\{r/\lambda_c + \lvert\tau\rvert/\tau_c\}}$	$\frac{1}{\{1+(\omega\tau_c)^2\}\{1+(k\lambda_c)^2\}^3}$	$\frac{8}{3\pi^2}$	$\{1+(k\lambda_c)^2\}\hat{h}(k,\omega)$	$\frac{1}{\pi^3}$

$f(r,\tau)$ = longitudinal correlation function

$\hat{f}(k,\omega)$ = Fourier transform of longitudinal correlation function

$\hat{f}_o$ = $\hat{f}(0,0)$

$E(k,\omega)$ = energy spectrum function = $C\cdot(\overline{u^2}\lambda_c^5\tau_c)\cdot\hat{h}(k,\omega)\cdot k^4$

[See equations (3.59)-(3.66) and (3.75)-(3.76).]

3.6 Isotropic turbulence and decaying mean fields - initial condition *II*

3.6.1 The nature of the eigenvalue for oscillating mean fields

We may now use (3.69) and (3.73) to determine the nature of the "eigenvalue" η/AK^5 in (3.47) for the cases of "Gaussian" and "Gaussian-exponential" turbulence. Comparing the two equations, we see that

$$AK^5/\eta = C \cdot \frac{\overline{u^2}\lambda_c^5 \tau_c}{\eta} \cdot K^5 = C \cdot \left\{ \frac{\overline{u^2}\lambda_c^2}{\eta^2} \right\} (\lambda_c K)^5 (\eta \tau_c/\lambda_c^2)$$

$$= C \cdot \{(R_m')^2/q\} (\lambda_c K)^5$$

$$= C \cdot (R_m')^2 (\lambda_c K)^3 (\eta K^2 \tau_c) \qquad (3.75)$$

where we have made use of the definitions (2.68) and (2.69), and the identification

$$\mathcal{U}' \equiv \sqrt{\overline{u^2}} \qquad (3.76)$$

C is a constant determined by the form of $f(r,\tau)$.

It may be seen from Table 12 that (3.75) is a general expression applying to any type of stationary, homogeneous, isotropic turbulence. Values of C are given in the table for "Gaussian", "Gaussian-exponential", "exponential-Gaussian", and "exponential" turbulence.

The quantity $\lambda_c K$ in (3.75) can be interpreted as a ratio of length scales. The wavelength of the mean field

is

$$L = 2\pi/K \tag{3.77}$$

so that

$$\lambda_c K = 2\pi\cdot(\lambda_c/L) \tag{3.78}$$

Similarly, the quantity $\eta K^2 \tau_c$ can be interpreted as a ratio of time scales. In the absence of turbulence, the mean field $\bar{\underset{\sim}{B}}$ will decay as $\exp[-\eta K^2 t]$, so that the mean field decay time is

$$T_d = 1/\eta K^2 \tag{3.79}$$

Therefore,

$$\eta K^2 \tau_c = \tau_c/T_d \tag{3.80}$$

and the eigenvalue η/AK^5 can be expressed as

$$(\eta/AK^5)^{-1} = (2\pi)^3\cdot C\cdot(R_m')^2\cdot(\lambda_c/L)^3\cdot(\tau_c/T_d)$$

3.6.2 Non-oscillatory mean fields - the dispersion relation and the effective magnetic diffusivity

We may now examine the solutions of (3.39) more closely. Consider first the case in which $\mathrm{Re}\ \Omega = 0$. Then by (3.15), (3.44), (3.45), and (3.75) the mean field dispersion relation has the form

$$\mathrm{Im}\,\frac{\Omega}{\eta K^2} - 1$$
$$= \frac{1}{4} C \frac{(R_m')^2}{q} (\lambda_c K)^5 \int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\xi \; \hat{h}(K\xi, \eta K^2 \nu) \,\mathrm{Re}\,\Theta\{\xi, \nu; \mathrm{Im}\,\frac{\Omega}{\eta K^2}\} \tag{3.83}$$

The parameter $\mathrm{Im}\ \Omega/\eta K^2$ may be interpreted in terms of an *effective magnetic diffusivity*, η_{eff} . When no turbulence is present, the solution of (3.83) is

$$\mathrm{Im}\,\Omega = \eta K^2 \tag{3.84}$$

and the mean field decays as $\exp[-\eta K^2 t]$. When there is turbulence present, the mean field decays more rapidly, with the exponential factor now being $\exp[-\mathrm{Im}\Omega \cdot t]$. We may therefore define

$$\mathrm{Im}\,\Omega \equiv \eta_{\mathrm{eff}} K^2 \tag{3.85}$$

by analogy with (3.84), so that

$$\mathrm{Im}\,\Omega/\eta K^2 = \eta_{\mathrm{eff}}/\eta \tag{3.86}$$

Equation (3.83) may be rewritten in the form

$$(R_m')^2 = \frac{4q}{C\cdot(\lambda_c K)^5} \{\eta_{\mathrm{eff}}/\eta - 1\} \Big\{ \int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\xi \; \hat{h}(K\xi, \eta K^2 \nu) \cdot \cdot \mathrm{Re}\,\Theta(\xi, \nu; \eta_{\mathrm{eff}}/\eta) \Big\}^{-1} \tag{3.87}$$

specifying the *magnetic Reynolds number* of the turbulence, R_m' , as a function of η_{eff}/η , q , $\lambda_c K$, and $\eta K^2 \tau_c$. In principle, (3.87) can be inverted to give η_{eff}/η as a function of R_m' , q , $\lambda_c K$, and $\eta K^2 \tau_c$.

3.6.3 Insensitivity of the effective diffusivity to initial conditions when $\lambda_c K$ is small

In order to establish a correspondence between (3.83) and the results obtained using the Rädler expansion technique, we must investigate the limiting behaviour of (3.83) as $\lambda_c K \to 0$. Making a change of variable in the integral,

$$\frac{4}{C(R_m')^2}\left\{\eta_{eff}/\eta - 1\right\}$$

$$= (\lambda_c K)^2 \int_{-\infty}^{\infty} d\nu \int_0^{\infty} \hat{h}\left\{\xi/\lambda_c , \nu/\tau_c\right\} \mathrm{Re}\,\Theta\left\{\xi/\lambda_c K , \nu/\eta K^2\tau_c ; \eta_{eff}/\eta\right\} d\xi \tag{3.88}$$

From (3.42) and *Figures 1 and 2* we can see that the dominant contribution to the integral in (3.88) will come from the values of $\mathrm{Re}\,\Theta\,(\xi,\nu;p)$ at large ξ , since $\mathrm{Re}\,\Theta \to \infty$ as $\xi \to \infty$. From (3.42) we have

$$\mathrm{Re}\,\Theta\left\{\xi/\lambda_c K , \nu/\eta K^2\tau_c ; p\right\} = \mathrm{Re}\,\Theta\left\{\xi/\lambda_c K , q\nu/(\lambda_c K)^2 ; p\right\}$$

$$= \frac{2\xi^4}{(\lambda_c K)^2}\int_0^{\pi} \frac{\xi^2 + 2(\lambda_c K)\xi\cos\theta + (\lambda_c K)^2(1-p)}{\left[\xi^2 + 2(\lambda_c K)\xi\cos\theta + (\lambda_c K)^2(1-p)\right]^2 + \left[\nu q\right]^2}\sin^3\theta\, d\theta \tag{3.89}$$

Expanding the integrand in (3.89) as a power series in $\lambda_c K$ and integrating term by term,

$$\mathrm{Re}\,\Theta\{\xi/\lambda_c K,\ \nu/\eta K^2\tau_c;\ P\}$$

$$= \frac{8}{3(\lambda_c K)^2}\ \frac{\xi^6}{\{\xi^4 + (\nu q)^2\}}\ \cdot$$

$$\cdot\Big\{1 + \frac{(\lambda_c K)^2}{\xi^2[\xi^4 + (\nu q)^2]}\Big[\frac{1}{5}\xi^4(3\xi^4 - \nu^2 q^2) + (p-1)(\xi^8 - \nu^4 q^4)\Big]$$

$$+ \cdots \Big\} \qquad (3.90)$$

Substituting (3.90) into (3.88) and dropping terms of order $(\lambda_c K)^2$,

$$\{\eta^{eff}/\eta - 1\} = \frac{2C(R_m')^2}{3}\int_{-\infty}^{\infty} d\nu \int_0^{\infty} \frac{\xi^6}{\xi^4 + (\nu q)^2}\,\hat{h}(\xi/\lambda_c, \nu/\tau_c)\, d\xi$$

$$+ \mathcal{O}\{(\lambda_c K)^2\} \qquad (3.91)$$

Equation (3.91) shows that when $(\lambda_c K)^2$ is small the effective diffusivity is *independent of the properties of the mean field*, and depends only on the properties of the turbulence. The equation can be shown to be identical to the result obtained by *Krause and Rädler (1971, equations 7.25, 7.39a, 7.43b)*, using initial condition *I* (3.34) rather than initial condition *II* (3.38). The effective diffusivity is therefore also *independent of the initial conditions on* $\underset{\sim}{B}'$ when $(\lambda_c K)^2$ is small.

3.6.4 Comparison of results for small $\lambda_C K$ with the results of *Krause and Rädler*

Applying (3.91) to turbulence of the Gaussian type (3.67)-(3.70), we obtain

$$\left\{\eta_{eff}/\eta - 1\right\} = \frac{(R_m')^2}{9\pi}\int_{-\infty}^{\infty} d\nu \int_0^{\infty} \frac{\xi^6}{\xi^4+(\nu q)^2}\, e^{-\frac{1}{2}(\xi^2+\nu^2)}\, d\xi + O(\lambda_c^2 K^2)$$

$$= \frac{(R_m')^2}{9}\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-(qt)^2/2}\, dt \int_0^{\infty} \xi^4 e^{-(t+\frac{1}{2})\xi^2} d\xi + O(\lambda_c^2 K^2)$$

$$= \frac{1}{3}(R_m')^2 \int_0^{\infty} e^{-(qt)^2/2} \frac{dt}{(2t+1)^{5/2}} + O(\lambda_c^2 K^2) \qquad (3.92)$$

making use of a number of standard integral formulae *(Gradshteyn and Ryzhik, 1965, equations 3.466.1, 8.252.6, and 3.461.2).*

Similarly, for turbulence of the Gaussian-exponential type (3.71)-(3.74),

$$\left\{\eta_{eff}/\eta - 1\right\} = \frac{(R_m')^2}{9\pi}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} d\nu \int_0^{\infty} \frac{\xi^6}{\left\{\xi^4+(\nu q)^2\right\}} \frac{e^{-\xi^2/2}}{(1-\nu^2)}\, d\xi + O(\lambda_c^2 K^2)$$

$$= \frac{1}{9}(R_m')^2\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\xi^4}{\xi^2+q}\, e^{-\xi^2/2}\, d\xi + O(\lambda_c^2 K^2)$$

$$= \frac{1}{9}(R_m')^2\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-qt}\, dt \int_0^{\infty} \xi^4 e^{-(t+\frac{1}{2})\xi^2} d\xi + O(\lambda_c^2 K^2)$$

$$= \frac{1}{3}(R_m')^2 \int_0^{\infty} e^{-qt} \frac{dt}{(2t+1)^{5/2}} + O(\lambda_c^2 K^2) \qquad (3.93)$$

(Gradshteyn and Ryzhik, 1965, equations 3.264.2 and 3.461.2)

The final expressions in (3.92) and (3.93) have been written in a form closely similar to that used by *Krause and Rädler (1971, pp. 67-70)* to facilitate comparison. In *Figure 3*, $9[(\eta_{eff}/\eta)-1]/R_m'$ has been plotted as a function of q for $\lambda_c K = 0$, making use of (3.92) and (3.93). The upper curve corresponds to (3.92) and the lower curve to (3.93).

It should be noted that *equation 7.34, p. 69* of *Krause and Rädler (1971)* is in error. For Gaussian turbulence, the expression should read

$$\beta^{(\lambda\nu)} = \frac{\lambda_c^{2\lambda}\,\tau_c^{\nu+1}}{3\cdot 2^{\lambda}\cdot\lambda!\,\nu!}\,(q/2)^{5/2}\int_0^{\infty}\frac{x^{\nu+\lambda}\,e^{-x^2/2}}{(x+q/2)^{\lambda+5/2}}\,dx \tag{3.94}$$

while for Gaussian-exponential turbulence,

$$\beta^{(\lambda\nu)} = \frac{\lambda_c^{2\lambda}\,\tau_c^{\nu+1}}{3\cdot 2^{\lambda}\cdot\lambda!\,\nu!}\,(q/2)^{5/2}\int_0^{\infty}\frac{x^{\nu+\lambda}\,e^{-x}}{(x+q/2)^{\lambda+5/2}}\,dx \tag{3.95}$$

The $\beta^{(\lambda\nu)}$ are coefficients in an expansion of the dispersion relation (3.33) for initial condition *I*.

$$i\Omega + \eta K^2 = -\overline{u^2}K^2\sum_{\nu,\lambda=0}^{\infty}(-1)^{\nu+\lambda}\beta^{(\lambda\nu)}(i\Omega)^{\nu}K^{2\lambda} \tag{3.96}$$

Thus, to a first approximation when $\mathrm{Re}\,\Omega = 0$,

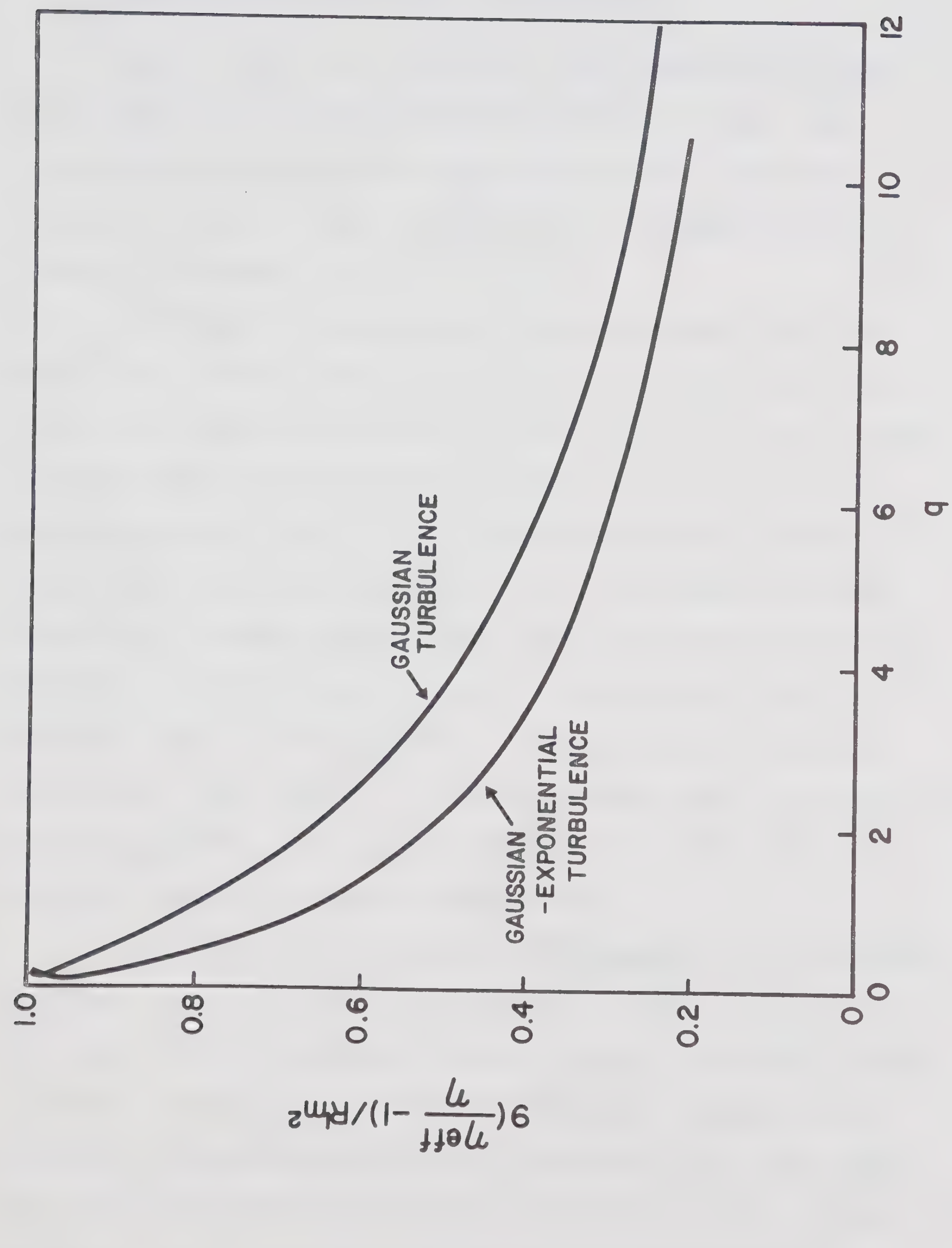
$$\eta_{eff}/\eta - 1 = \frac{\overline{u^2}}{\eta}\beta^{(00)} + \cdots \tag{3.96'}$$

and this equation may be compared with (3.92) and (3.93).

<u>Figure 3.</u> $9\{[\eta_{eff}/\eta]-1\}/\{R'_m\}^2$ as a function of q when $\lambda_c/L = 0$.

Upper curve: Gaussian turbulence
(see equation 3.92)

Lower curve: Gaussian-exponential turbulence
(see equation 3.93)


GAUSSIAN TURBULENCE
GAUSSIAN -EXPONENTIAL TURBULENCE
q
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
1.0
$9(\frac{\eta_{eff}}{\eta} - 1)/Rm^2$

3.6.5 <u>The effects of initial conditions and mean field nonuniformity when $\lambda_c K$ is not small</u>

When $(\lambda_c K)^2$ is not small, the properties of the mean field $\bar{\underset{\sim}{B}}$ and the initial conditions on $\underset{\sim}{B}'$ have significant effects on the solutions of the dispersion relation (3.83). These effects are illustrated by the curves in *Figures 4 and 5.*

In *Figure 4* the magnetic Reynolds number R_m' is shown as a function of $[(\eta_{eff}/\eta)-1]$ for Gaussian turbulence, for several values of q and $\lambda_c K$. It is interesting to note that the dispersion relation (3.83) has no solutions when $[(\eta_{eff}/\eta)-1]$ is greater than a critical value which depends mainly on $\lambda_c K$. The reason for this behaviour is immediately evident from equations (3.83) and (3.42), and from *Figures 1 and 2.* Re $\Theta(\xi,\nu;\eta_{eff}/\eta)$ is negative for $\xi < \sqrt{(\eta_{eff}/\eta)-1}$. Thus if $\lambda_c K$ is large enough for $\hat{h}(K\xi, \eta K^2\nu)$ to be effectively zero at values of $\xi > \sqrt{(\eta_{eff}/\eta)-1}$, the right hand side of (3.83) will be negative, making a solution impossible.

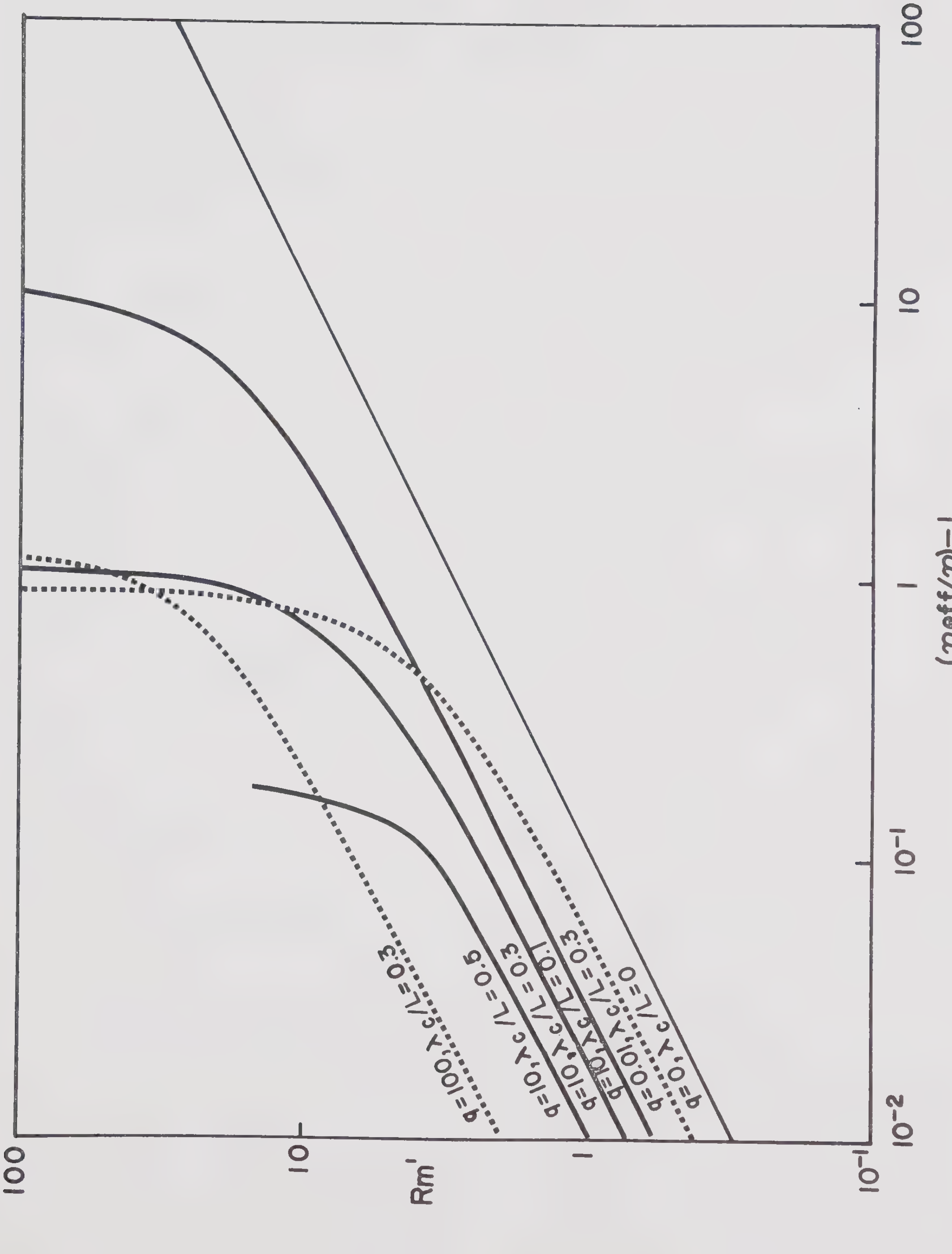
In *Figure 5* the function $R_m'^2/\{(\eta_{eff}/\eta)-1\}$ is plotted against $\lambda_c/L = \lambda_c K/2\pi$ for various q when $\eta_{eff}/\eta = 1.1$. The upper curves correspond to the initial condition (3.38) *(initial condition II)*, and the turbulence considered is Gaussian. Gaussian-exponential turbulence will have qualitatively the same behaviour, but the values of the intercepts at $\lambda_c K = 0$ will have a different dependence on q.

This difference is clearly illustrated in *Figure 3*.

The lower curves in *Figure 5* correspond to the initial condition (3.34) *(initial condition I)*, which will be considered in the next section.

<u>Figure 4</u>. R'_m as a function of $[\eta_{eff}/\eta]-1$ for several values of λ_c/L and q. *(Initial condition I.)*

The plot shows values for Gaussian turbulence, determined from equation (3.87) with $\hat{h}(k,\omega)$ given by equation (3.70).



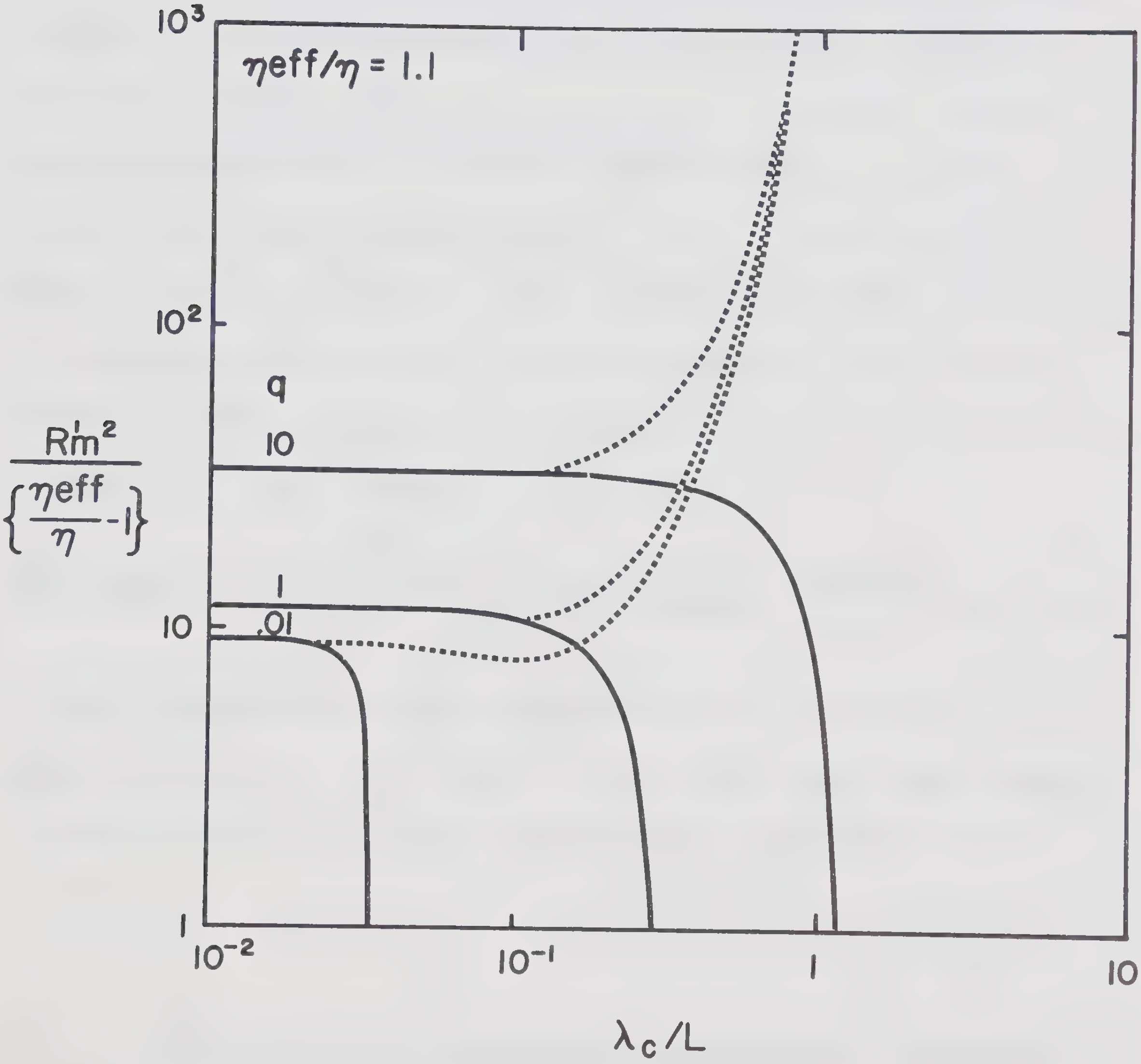
Rm'
100
10
1
10^{-1}
10^{-2}
10^{-1}
1
10
100
$(\eta_{eff}/\eta)-1$
$q=100, \lambda_c/L=0.3$
$q=10, \lambda_c/L=0.5$
$q=10, \lambda_c/L=0.3$
$q=10, \lambda_c/L=0.1$
$q=0.01, \lambda_c/L=0.3$
$q=0, \lambda_c/L=0$

Figure 5. $\{R'_m\}^2/\{[\eta_{eff}/\eta]-1\}$ as a function of λ_c/L for several values of q when $\eta_{eff} = 1.1\,\eta$.

The plot shows values for Gaussian turbulence, determined from equation (3.87) with $\hat{h}(k,\omega)$ given by equation (3.70).

Solid curves: initial condition *I*

Dashed curves: initial condition *II*



$\eta eff/\eta = 1.1$
q
10
1
.01
$\frac{R'm^2}{\left\{\frac{\eta eff}{\eta}-1\right\}}$
10^3
10^2
10
1
10^{-2}
10^{-1}
1
10
λ_c/L

3.7 PT-invariant turbulence and decaying mean fields - initial condition *I*

3.7.1 The mean field dispersion relation

Let us now consider the solution of the dispersion relation (3.35), corresponding to the initial condition (3.34) *(initial condition I)* on $\underset{\sim}{B}'$. As noted above, this solution does not differ significantly from the solution (3.91) corresponding to *initial condition II*, when $(\lambda_c K)^2$ is small. From (3.35) we see that the choice of initial condition has removed the pole (3.41) which occurs on the integration contour in (3.40). The dispersion relation is best written in the form (3.35')

$$i\Omega + \eta K^2 = -K_j \int_0^{t-t_0} e^{-i\Omega\tau} d\tau \int_{\underset{\sim}{k}} (\underset{\sim}{k}+\underset{\sim}{K})_p \hat{R}_{jp}(\underset{\sim}{k};\tau)\, e^{-\eta(\underset{\sim}{k}+\underset{\sim}{K})^2\tau} d\underset{\sim}{k} \tag{3.35'}$$

It may be noted that the spatial Fourier transforms of the quantities R_{ij} , f , and g bear the same relationship to one another as do the full Fourier transforms (3.51)-(3.53).

3.7.2 The mean field dispersion relation for isotropic turbulence in an incompressible fluid

For homogeneous, stationary, isotropic turbulence in an incompressible fluid, (3.35') reduces to

$$\{i\Omega + \eta K^2\} =$$

$$= \frac{\pi}{3}\overline{u^2}K^2\int_0^{t-t_o} e^{-\{i\Omega+\eta K^2\}\tau}\,d\tau \;\cdot$$

$$\cdot \int_0^\infty k^3 \frac{\partial \hat{f}(k;\tau)}{\partial k} e^{-\eta k^2\tau} dk \int_0^\pi \sin^3\theta\, e^{-2\eta k K\tau\cos\theta}\,d\theta$$

$$= \frac{\pi}{3}\overline{u^2}K^2\int_0^{t-t_o} e^{-\{i\Omega+\eta K^2\}\tau}\,d\tau \;\cdot$$

$$\cdot \frac{1}{(\eta K\tau)^2}\int_0^\infty k \frac{\partial \hat{f}(k;\tau)}{\partial k} e^{-\eta k^2\tau}\left\{\cosh 2\eta k K\tau - \frac{\sinh 2\eta k K\tau}{2\eta k K\tau}\right\} dk \tag{3.97}$$

For turbulence with a Gaussian spatial correlation, like (3.67) and (3.71),

$$\hat{f}(k;\tau) = \frac{\lambda_c^3}{(2\pi)^{3/2}} e^{-(\lambda_c k)^2/2}\,\psi(\tau)$$

$$k\frac{\partial \hat{f}(k;\tau)}{\partial k} = -\frac{\lambda_c^5}{(2\pi)^{3/2}} k^2 e^{-(\lambda_c k)^2/2}\,\psi(\tau) \tag{3.98}$$

so that (3.97) becomes

$$i\Omega + \eta K^2 = -\frac{\overline{u^2}\lambda_c^5}{3}K^2\int_0^{t-t_o} e^{-\left\{i\Omega + \frac{\eta(\lambda_c K)^2}{\lambda_c^2+2\eta\tau}\right\}\tau} \frac{\psi(\tau)\,d\tau}{\{\lambda_c^2+2\eta\tau\}^{5/2}} \tag{3.99}$$

after some reduction. A similar dispersion relation was developed by *Lerche (1971a,b)* and *Lerche and Low (1971)*, in the limit as $(t-t_o) \to \infty$.

3.7.3 Time dependence of the mean field dispersion relation for different types of turbulence

Taking the derivative of the right hand side of (3.99) with respect to time, we see that

$$\frac{\partial}{\partial t}\left\{i\Omega + \eta K^2\right\} =$$

$$= -\frac{1}{3}\overline{u^2}\lambda_c^5 K^2 e^{-\left\{i\Omega + \frac{\eta(\lambda_c K)^2}{\lambda_c^2 + 2\eta(t-t_o)}\right\}(t-t_o)} \frac{\psi(t-t_o)}{\left\{\lambda_c^2 + 2\eta(t-t_o)\right\}^{5/2}}$$

$$\xrightarrow[(t-t_o)\to\infty]{} \quad -\frac{1}{3}\overline{u^2}\lambda_c^5 K^2 e^{-i\Omega(t-t_o)} \frac{\psi(t-t_o)}{\left\{2\eta(t-t_o)\right\}^{5/2}} \tag{3.100}$$

Clearly, (3.100) will diverge as $(t-t_o) \to \infty$ unless

$$\lim_{t\to\infty} e^{\mathrm{Im}\,\Omega(t-t_o)}\,\psi(t-t_o)\,(t-t_o)^{-5/2} = 0 \tag{3.101}$$

For turbulence with a Gaussian time correlation, (3.67), the condition (3.101) is satisfied for all values of $\mathrm{Im}\,\Omega$; however, for turbulence with an exponential time correlation, (3.71), condition (3.101) is only satisfied if

$$\tau_c\,\mathrm{Im}\,\Omega \leq 1 \tag{3.102}$$

For turbulence with a power-law time correlation, (3.101) is *never* satisfied when $\mathrm{Im}\,\Omega = \text{constant} > 0$, and the dispersion relation (3.99) has no meaning.

The time dependence of the dispersion relation (3.99) can be regarded as an indication of the way in which the turbulence works to "build" a fluctuating field $\underset{\sim}{B}'$ of

the form required by the induction equation, at the expense of the mean field $\bar{B}$. If the memory of the turbulence is "too long" - as in the case of a power-law correlation - the effect of the initial, inappropriate field $B'(x,t_o)$ will persist, and make it impossible for the turbulence to maintain even a decaying mean field of the type (3.14). However, if the memory of the turbulence is "short" - as with a Gaussian time correlation - the effects of the initial, inappropriate B'-field will die away rapidly enough for the dispersion relation (3.99) to *stabilize*. For Gaussian turbulence (3.67), this stabilization will take place fairly quickly - typically after a few correlation times. We shall therefore restrict attention in what follows to the case of Gaussian turbulence.

3.7.4 The mean field dispersion relation for Gaussian isotropic turbulence - non-oscillatory mean fields

For Gaussian turbulence (3.67), the time correlation function is

$$\psi(\tau) = e^{-\tau^2/2\tau_c^2} \tag{3.102}$$

and the dispersion relation (3.99) becomes

$$i\Omega + \eta K^2 = -\frac{1}{3}\overline{u^2}\lambda_c^5 K^2 \int_0^{t-t_o} e^{-\tau^2/2\tau_c^2 - \left\{i\Omega + \frac{\eta(\lambda_c K)^2}{\lambda_c^2 + 2\eta\tau}\right\}\tau} \frac{d\tau}{\{\lambda_c^2 + 2\eta\tau\}^{5/2}} \tag{3.103}$$

Equation (3.103) can be written in a number of alternative forms by making appropriate changes of variable. The most useful of these forms are:

$$i\Omega/\eta K^2 + 1$$

$$= -\frac{1}{3}(R_m')^2 \int_0^{\frac{(t-t_o)}{q\tau_c}} e^{-\frac{1}{2}q^2x^2 - (\lambda_c K)^2\left\{\frac{i\Omega}{\eta K^2} + \frac{1}{1+2x}\right\}x} \frac{dx}{(1+2x)^{5/2}} \tag{3.103'}$$

$$= -\frac{1}{3}\frac{(R_m')^2}{q} \int_0^{\frac{(t-t_o)}{\tau_c}} e^{-\frac{1}{2}x^2 - (\eta K^2\tau_c)\left\{\frac{i\Omega}{\eta K^2} + \frac{1}{1+2x/q}\right\}x} \frac{dx}{(1+2x/q)^{5/2}} \tag{3.103"}$$

When $\mathrm{Re}\ \Omega = 0$, these equations reduce to:

$$\mathrm{Im}\,\Omega/\eta K^2 - 1$$

$$= \frac{1}{3}(R_m')^2 \int_0^{\frac{(t-t_o)}{q\tau_c}} e^{-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\mathrm{Im}\frac{\Omega}{\eta K^2} - \frac{1}{1+2x}\right\}x} \frac{dx}{(1+2x)^{5/2}} \tag{3.104}$$

$$= \frac{1}{3}\frac{(R_m')^2}{q} \int_0^{\frac{(t-t_o)}{\tau_c}} e^{-\frac{1}{2}x^2 + \frac{(\lambda_c K)^2}{q}\left\{\mathrm{Im}\frac{\Omega}{\eta K^2} - \frac{1}{1+2x/q}\right\}x} \frac{dx}{(1+2x/q)^{5/2}} \tag{3.104'}$$

It will be noted that (3.103)-(3.104') are *integral equations* for $\Omega(t-t_o)$. When $\lambda_c K = 0$ the integrands no longer depend on $\Omega(t-t_o)$ and in the limit $(t-t_o) \to \infty$ (3.104) reduces to the form (3.92) specified by the *Rädler* expansion (3.96). When $\lambda_c K \neq 0$, on the other hand, $\mathrm{Im}\ \Omega/\eta K^2$ satisfies the inequality

$$(R_m')^2 J_1 \le \mathrm{Im}\frac{\Omega}{\eta K^2} - 1 \le (R_m')^2 J_2 \tag{3.105}$$

where

$$J_1(q, \lambda_c K; T) \equiv \frac{1}{3}\int_0^{T/q\tau_c} e^{-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\frac{2x^2}{1+2x}\right\}} \frac{dx}{(1+2x)^{5/2}} \tag{3.106}$$

$$J_2(q, \lambda_c K; T) \equiv \frac{1}{3}\int_0^{T/q\tau_c} e^{-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\mathrm{Im}\frac{\Omega(T)}{\eta K^2} - \frac{1}{1+2x}\right\}x} \frac{dx}{(1+2x)^{5/2}} \tag{3.107}$$

(3.106) and (3.107) have been obtained by setting $\mathrm{Im}\ \Omega/\eta K^2 = 1$ and $\mathrm{Im}\ \Omega/\eta K^2 = [\mathrm{Im}\ \Omega/\eta K^2]_{t=t_o+T}$ respectively on the right hand side of equation (3.104).

It follows from (3.104) and (3.105) that

$$\frac{\left\{\mathrm{Im}\,\Omega/\eta K^2 - 1\right\}}{J_2(q, \lambda_c K; T)} \leq (R_m')^2 \leq \frac{\left\{\mathrm{Im}\,\Omega/\eta K^2 - 1\right\}}{J_1(q, \lambda_c K; T)} \tag{3.108}$$

$$\frac{\left\{\eta_{eff}/\eta - 1\right\}}{J_2(q, \lambda_c K; \infty)} \leq (R_m')^2 \leq \frac{\left\{\eta_{eff}/\eta - 1\right\}}{J_1(q, \lambda_c K; \infty)} \tag{3.108'}$$

The two sides of the inequality (3.108') are plotted in *Figure 6* for particular values of $\lambda_c K$ and q. In (3.108') we have used the definition

$$\eta_{eff}/\eta \equiv \lim_{(t-t_o)\to\infty} \mathrm{Im}\,\Omega/\eta K^2 \tag{3.109}$$

Figure 6. R'_m as a function of $[\eta_{eff}/\eta]-1$ for $\lambda_c/L = 0.3$ and $q = 10$.

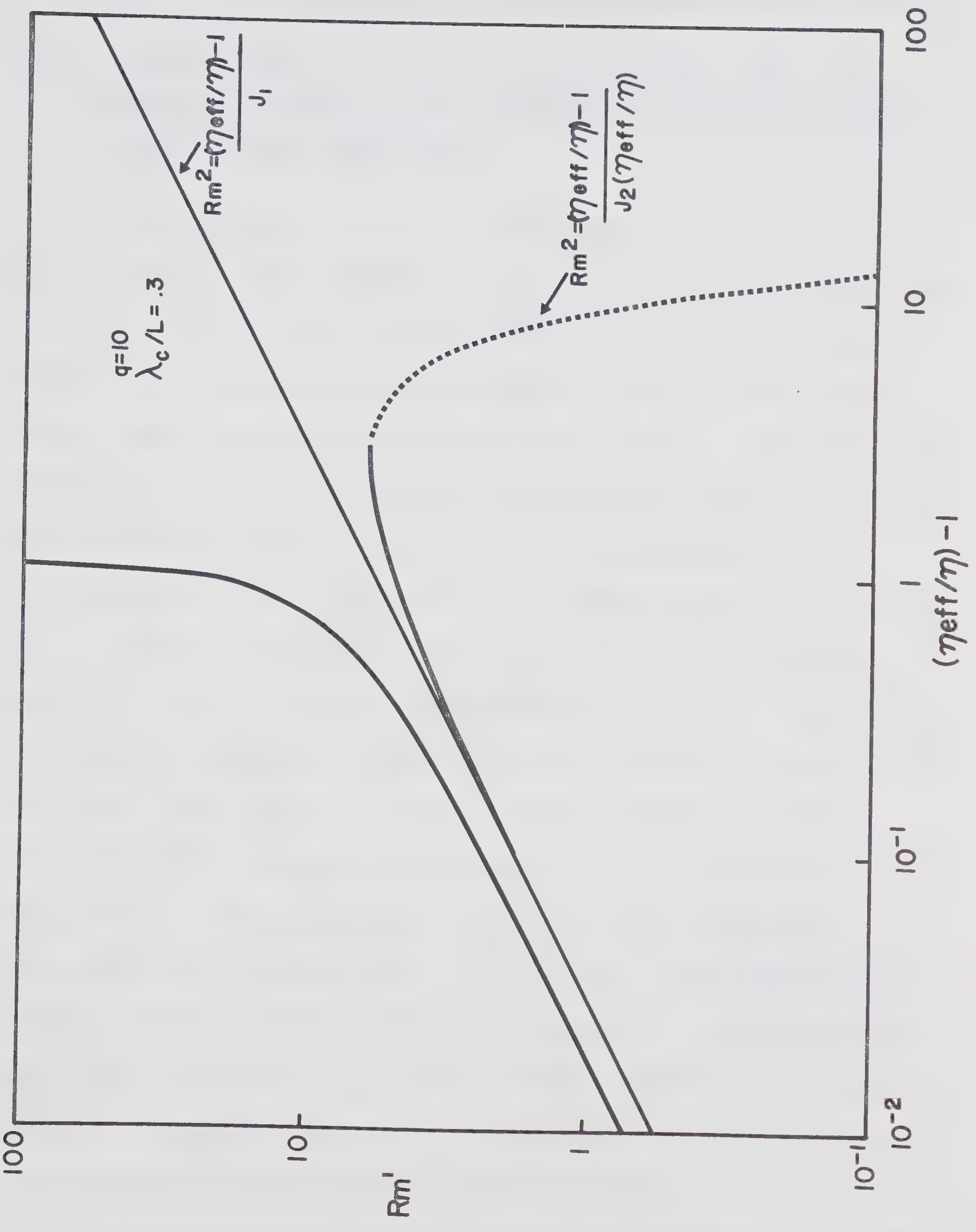
The plot shows values for Gaussian turbulence.

Lowermost curve: $\{R'_m\}^2 = \{[\eta_{eff}/\eta]-1\}/J_2$
(see left hand side of 3.108')

Middle curve: $\{R'_m\}^2 = \{[\eta_{eff}/\eta]-1\}/J_1$
(see right hand side of 3.108')

Uppermost curve: R'_m determined from equation (3.87) with $\hat{h}(k,\omega)$ given by equation (3.70).

The two lower curves correspond to initial condition *I*, while the uppermost curve corresponds to initial condition *II*.


q=10
$\lambda_c/L = .3$
$Rm^2 = \frac{(\eta_{eff}/\eta)-1}{J_1}$
$Rm^2 = \frac{(\eta_{eff}/\eta)-1}{J_2(\eta_{eff}/\eta)}$
Rm'
100
10
1
10^{-1}
10^{-2}
10^{-1}
1
10
100
$(\eta_{eff}/\eta)-1$

3.8 Isotropic turbulence and decaying, non-oscillatory mean fields - comparison of initial conditions *I* and *II*

3.8.1 Relationship between mean field decay rate and turbulent magnetic Reynolds number - effect of mean field initial conditions

From *Figure 6* it can be seen that the true value of R_m' is closely approximated by both sides of the inequality (3.108'), up to fairly large values of the ratio η_{eff}/η. (This ratio may be taken as a measure of the mean field decay rate, by virtue of [3.79] and [3.86].) Moreover, the solutions for the two initial conditions, *I* and *II*, do not differ greatly when $(\eta_{eff}/\eta)-1$ is small, even for the comparatively large value of $\lambda_c K$ considered $[\lambda_c/L = 0.3]$.

The most striking difference between the lowest curve in *Figure 6*, which corresponds to the left hand side of (3.108'), and the other two curves, which correspond to the right hand side of (3.108') and to (3.87), is the occurrence of a *maximum* in the plot of R_m' against $(\eta_{eff}/\eta)-1$. This behaviour is due to the nonlinear dependence of $J_2(q,\lambda_c K;\infty)$ on η_{eff}/η. When $\frac{(\lambda_c K)^2}{q}\frac{\eta_{eff}}{\eta}$ is small, J_2 is *independent* of η_{eff}/η. However, when $\frac{(\lambda_c K)^2}{q}\frac{\eta_{eff}}{\eta}$ is large, J_2 has a roughly exponential dependence on η_{eff}/η, causing a rapid decrease in the value of R_m' given by the left hand side of (3.108').

It may be seen from (3.105) that points on the curve

beyond the maximum are of no interest. If we assume for the moment that $\mathrm{Im}\,\Omega(T)/\eta K^2$ is *defined* as a function of $T = t-t_0$ by the right hand side of (3.105), and the problem is regarded as one of switching on a fully developed turbulent velocity field at time $T = 0$ and observing the subsequent behaviour of a mean field $\overline{\underset{\sim}{B}}$ present in the fluid at the initial instant, we see that the decay rate of the mean field at large T depends on the initial decay rate,

$$(\mathcal{I}m\,\Omega)_0 \equiv \mathcal{I}m\,\Omega(T)\Big|_{T=t-t_0=0} \tag{3.110}$$

Expanding (3.107) as a power series in T about $T = 0$,

$$\mathcal{I}m\,\Omega(T)/\eta K^2 - 1 = \frac{(R_m')^2}{3q\tau_c}\left\{e^{(\lambda_c K)^2(T/q\tau_c)(\mathcal{I}m\,\Omega)/\eta K^2}\right\}_{T=0}\cdot T + \cdots \tag{3.111}$$

we see that only two initial decay rates are possible:

$$(\mathcal{I}m\,\Omega)_0 = \eta K^2 \tag{3.112a}$$

$$(\mathcal{I}m\,\Omega)_0 = \infty \tag{3.112b}$$

The first of these values corresponds to a mean field decaying at the normal diffusive rate in a stationary conductor. The second, on the other hand, is completely unphysical. Detailed numerical study shows that points to the right of the maximum on the lowermost curve in *Figure 6* correspond to the unphysical condition, so that these points may be disregarded. *(See Appendix 3, section A.3.1 for details of numerical techniques.)*

Since the mean field initial conditions (3.112) apply to equation (3.104) as well as to the right hand side of (3.105), it is to be expected that a true plot of R_m' against $(\eta_{eff}/\eta)-1$ will also exhibit a maximum of the type shown in *Figure 6*. Further justification for this statement can be obtained by examining the derivatives of (3.104) and (3.105) with respect to $(R_m')^2$. From (3.105) we have

$$\frac{\partial}{\partial(R_m'^2)}\left\{\mathcal{J}m\,\Omega/\eta K^2\right\} =$$

$$= \frac{\left\{\mathcal{J}m\,\Omega/\eta K^2 - 1\right\}}{(R_m')^2}\cdot \tag{3.113}$$

$$\cdot\left\{1-\tfrac{1}{3}(R_m')^2(\lambda_c K)^2\int_0^{T/q\tau_c} x\, e^{-\frac{1}{2}q^2x^2+(\lambda_c K)^2\left\{\mathcal{J}m\frac{\Omega}{\eta K^2}-\frac{1}{1+2x}\right\}x}\frac{dx}{(1+2x)^{5/2}}\right\}^{-1}$$

while from (3.104),

$$\frac{\partial}{\partial(R_m'^2)}\left\{\mathcal{J}m\,\Omega/\eta K^2\right\} =$$

$$= \frac{\left\{\mathcal{J}m\,\Omega/\eta K^2 - 1\right\}}{(R_m')^2}\cdot$$

$$\cdot\left\{1-\tfrac{1}{3}(R_m')^2(\lambda_c K)^2\int_0^{T/q\tau_c}\left[\frac{\frac{\partial}{\partial(R_m'^2)}\left\{\mathcal{J}m\,\Omega(x)/\eta K^2\right\}}{\frac{\partial}{\partial(R_m'^2)}\left\{\mathcal{J}m\,\Omega(T)/\eta K^2\right\}}\right]\cdot\right. \tag{3.114}$$

$$\left.\cdot\, x\, e^{-\frac{1}{2}q^2x^2+(\lambda_c K)^2\left\{\mathcal{J}m\frac{\Omega}{\eta K^2}-\frac{1}{1+2x}\right\}x}\frac{dx}{(1+2x)^{5/2}}\right\}^{-1}$$

The right hand side of (3.113) can be made to go to infinity by choosing $(R_m')^2$ sufficiently large. Similarly, in

(3.114) the ratio of derivatives in the integrand runs from 0 to 1 as x spans the range of integration, so that the integral will be finite for all $(R_m')^2$ and of the same general character as the integral in (3.113). The right hand side of (3.114) can therefore also be made to go to infinity by choosing $(R_m')^2$ sufficiently large. As might be expected from the inequality (3.105), the maximum for (3.104) will occur at a larger value of R_m' than the maximum of the curve plotted in *Figure 6*.

When the unphysical solutions of (3.104) and (3.105) are neglected - i.e. when points to the right of the maximum in the plot of R_m' against $(\eta_{eff}/\eta)-1$ are ignored - we have *cut-off values* of both η_{eff}/η and R_m' beyond which no useful solutions occur. This behaviour may be contrasted with that of the uppermost curve in *Figure 6*, which corresponds to *initial condition II*. In this curve there is a cut-off only in η_{eff}/η .

3.8.2 Range of validity of solutions obtained using the *Rädler* expansion technique

It is interesting to note that the relationship between $(R_m')^2$ and $(\eta_{eff}/\eta)-1$ in *Figure 6* remains linear nearly all the way out to the cut-off point. This indicates that the approximate solution (3.92), derived with the aid of the Rädler expansion technique, is useful over a wide

range of parameters. From (3.104') we see that the first term in the expansion may be expected to provide a good approximation as long as the increasing exponential factor in the integrand is negligibly different from unity over the range in which the Gaussian factor $e^{-x^2/2}$ has a significant amplitude. Imposing the condition that the increasing exponential differ from unity by less than 5%, and assuming that the Gaussian factor is effectively zero when $x = 2$, we obtain a condition for validity of the first term in the Rädler expansion:

$$(\eta K^2 \tau_c)\left\{\mathcal{I}m\,\frac{\Omega}{\eta K^2} - (1+4/q)^{-1}\right\} = \frac{(\lambda_c K)^2}{q}\left\{\mathcal{I}m\,\frac{\Omega}{\eta K^2} - (1+4/q)^{-1}\right\} \lesssim 0.025 \quad (3.115)$$

When this condition is not satisfied, higher-order terms in the expansion must be retained.

Taking limiting values of q in (3.115), we obtain the conditions

$$(q \to 0) \quad \frac{(\lambda_c K)^2}{q}\,\mathcal{I}m\,\frac{\Omega}{\eta K^2} = \tau_c\,\mathcal{I}m\,\Omega \lesssim 0.025 \quad (3.115')$$

$$(q \to \infty) \quad \frac{(\lambda_c K)^2}{q}\left\{\mathcal{I}m\,\frac{\Omega}{\eta K^2} - 1\right\} = \tau_c\left\{\mathcal{I}m\,\Omega - \eta K^2\right\} \lesssim 0.025 \quad (3.115'')$$

for the first term of the Rädler expansion to be a valid approximation. *Figure 5* indicates that (3.115") is applicable even for values of q as low as 0.01.

The condition (3.115) clearly applies only to the case of turbulence with a Gaussian time correlation. If the time correlation falls off more slowly, the first term of the Radler expansion will be valid over a much narrower

range of parameters.

3.8.3 The mean field decay rate at large times

The curves of *Figure 5* illustrate a second major difference between solutions corresponding to *initial condition I* and those corresponding to *initial condition II*. It is apparent that for large values of $\lambda_c K$, *initial condition I leads to a more rapid decay rate of the mean field at large times than does initial condition II*. This behaviour is illustrated schematically in *Figure 7*. Curve "a" corresponds to a mean field decaying in the absence of turbulence. Curve "b" corresponds to the case in which $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ are switched on together at time $t = t_o$ in a correlated manner. Curve "c" corresponds to the case in which $\underset{\sim}{u}'$ is switched on at the initial instant and $\underset{\sim}{B}'$ is allowed to develop.

It is clear from the slope discontinuity in *Figure 7* that the initial condition for curve "b" (i.e. *initial condition II*) is somewhat unphysical. However, the behaviour of curve "c" (*initial condition I*) is also open to question. The reason for the difference between the slopes of the two curves at large times can be seen from a comparison of equations (3.35) and (3.39), rewritten in the form

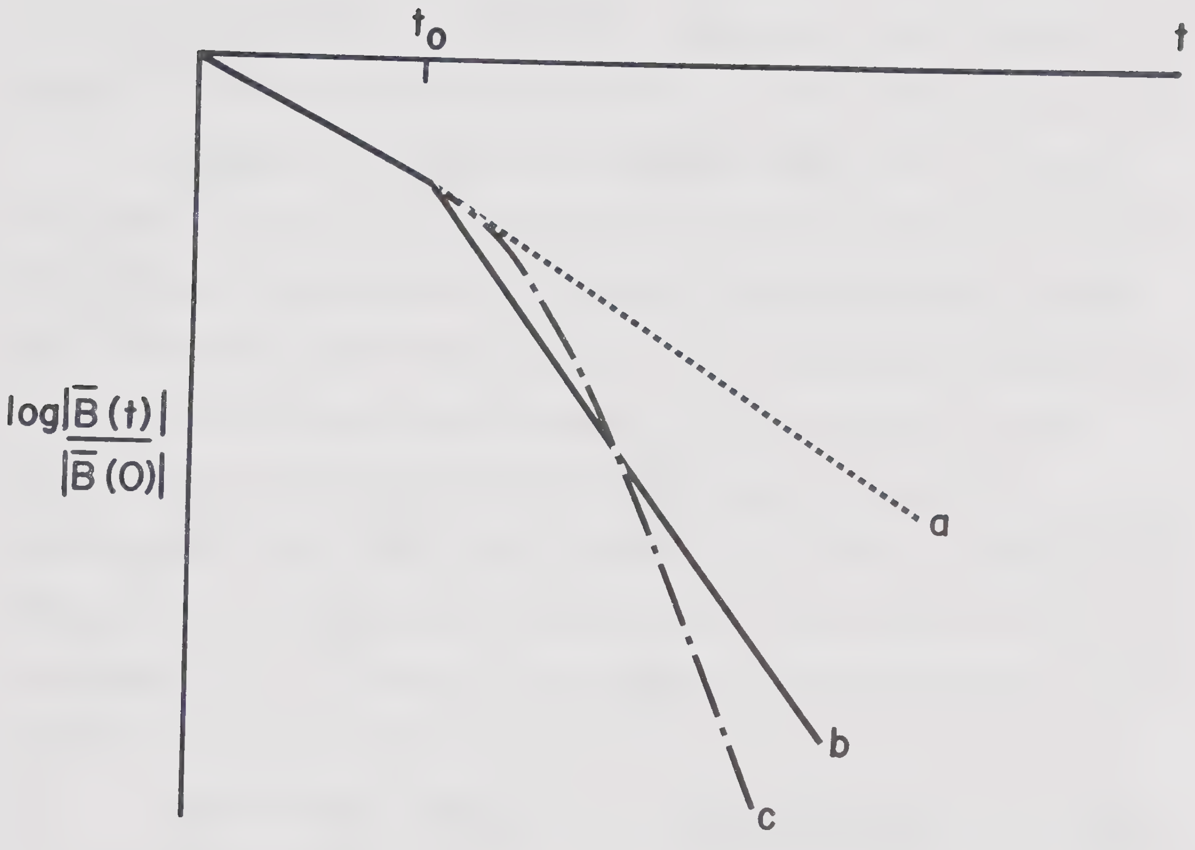
Figure 7. Decay of mean field amplitude with time.

(a) No turbulence

(b) Isotropic turbulence, with $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}\Big|_{t=t_o} \neq 0$

(c) Isotropic turbulence, with $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}\Big|_{t=t_o} = 0$

Curve (c) corresponds to *initial condition I*, and curve (b) corresponds to *initial condition II*.



t_0
t
$\log \frac{|\bar{B}(t)|}{|\bar{B}(0)|}$
a
b
c

$$i\Omega + \eta K^2 + I^{(1)} =$$

$$= - \int_{\substack{\underset{\sim}{k} \\ \eta(\underset{\sim}{k}+\underset{\sim}{K})^2 \leq \mathcal{I}m\,\Omega}} d\underset{\sim}{k} \int_{-\infty}^{\infty} d\tau \; K_j(\underset{\sim}{k}+\underset{\sim}{K})_p \, \hat{R}_{jp}(\underset{\sim}{k};\tau)\, e^{-\{i\Omega+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2\}\tau} \tag{3.116a}$$

$$i\Omega + \eta K^2 + I^{(1)} = 0 \tag{3.116b}$$

The right hand side of (3.116a) represents the limiting value of the time-dependent term in (3.35) as $(t-t_o) \to \infty$.

Clearly, the form of the spectrum tensor $\hat{R}_{jp}(\underset{\sim}{k};\tau)$ for small values of $\underset{\sim}{k}$ will be crucial to the evaluation of the right hand side of (3.116a). Unfortunately, it is at just these values of $\underset{\sim}{k}$ that our assumptions about the nature of the turbulence are most likely to be in error. For example, the assumption of isotropy at infinite separations will never be satisfied in a real, finite fluid. This difficulty does not arise in the evaluation of the integral $I^{(1)}$, which is dominated by contributions from values of $|\underset{\sim}{k}|$ near $2\pi/\lambda_c$.

For Gaussian turbulence in an incompressible fluid, (3.116a) reduces to

$$i\Omega + \eta K^2 + I^{(1)} =$$

$$= -\frac{1}{12\pi}\,\overline{u^2}\lambda_c^5\tau_c K^2 \int_{\eta(\underset{\sim}{k}+\underset{\sim}{K})^2 \leq \mathcal{I}m\,\Omega} d\underset{\sim}{k}\; k^2 e^{-(\lambda_c k)^2/2} \sin^2\theta \; e^{\frac{1}{2}\tau_c^2\{\mathcal{I}m\,\Omega - \eta(\underset{\sim}{k}+\underset{\sim}{K})^2\}^2}$$

when $\mathrm{Re}\,\Omega = 0$. If $\tau_c \mathrm{Im}\,\Omega \ll 1$, the second exponential factor can be set equal to unity over the entire range of

integration, and the equation becomes

$$i\Omega + \eta K^2 + I^{(1)} \approx$$

$$\underset{(\tau_c \mathscr{I}m\Omega \ll 1)}{\approx} -\frac{1}{6} u^2 \lambda_c^5 \tau_c K^2 \iint\limits_{\substack{k,\theta \\ \eta(k^2+K^2+2kK\cos\theta) \leq \mathscr{I}m\Omega}} k^4 e^{-(\lambda_c K)^2/2} \sin^3\theta \, dk \, d\theta$$

$$\lesssim \frac{2}{9} u^2 \lambda_c^5 \tau_c K^2 \int_0^{\sqrt{2\mathscr{I}m\Omega/\eta}} k^4 e^{-(\lambda_c k)^2/2} \, dk$$

$$\lesssim \frac{2}{9} (R_m')^2 q^{3/2} (\eta K^2) \int_0^{\sqrt{2\tau_c \mathscr{I}m\Omega}} t^4 e^{-qt^2/2} \, dt \qquad (3.117)$$

In order for the two initial conditions *I* and *II* to lead to the same result for this type of turbulence, we must therefore have

$$I^{(1)} \gg \frac{2}{9} (R_m')^2 q^{3/2} (\eta K^2) \int_0^{\sqrt{2\tau_c \mathscr{I}m\Omega}} t^4 e^{-qt^2/2} \, dt \qquad (3.118a)$$

and $$\tau_c \mathscr{I}m\Omega \ll 1 \qquad (3.118b)$$

The inequality (3.118a) is satisfied automatically when q is small, but it is not necessarily satisfied when q is large. It is apparent from (3.117) that the right hand side, which represents the difference between the two dispersion relations, is determined by the behaviour of the spectrum tensor at small values of $|\underset{\sim}{k}|$ whenever

$$q \ll 2(\tau_c \mathscr{I}m\Omega)^{-1} \qquad (3.119)$$

This condition is very likely to be satisfied - particularly

when (3.118b) is true and the first term in the Rädler expansion is a valid approximation.

On general grounds, we expect there to be less energy in the turbulence at small values of k than is implied by the use of the spectrum tensor (3.7), which is isotropic at $k = 0$. The right hand side of (3.116a) is therefore likely to be smaller than the value obtained using (3.7) - for example, the right hand side of (3.117) is probably an overestimate. It therefore appears that the slope of curve "c" *(Figure 7)* at large times may well be inaccurate when it differs markedly from the slope of curve "b".

3.8.4 Stabilization of the mean field decay rate, and loss of energy from the mean field

The initial, time-dependent portion of curve "c" in *Figure 7* is undoubtedly more realistic than the abrupt slope discontinuity shown in curve "b". It is only sensible to discuss the *stabilized* dispersion relation if the energy lost from the mean field during the time-dependent part of the decay is small. If T_1 is defined to be the time after which the mean field decay rate has reached its *stabilized* value, and T_2 is the time it takes the mean field amplitude to decay to $1/e$ of its value at time $t = t_o$, we may write

$$\frac{\mathcal{E}(T)}{\mathcal{E}_o} = \frac{\text{mean field energy density at } t = t_o + T}{\text{mean field energy density at } t = t_o}$$

$$= e^{-2\,\mathrm{Im}\,\Omega(T)\cdot T} \tag{3.120}$$

$$\frac{\mathcal{E}(T)}{\mathcal{E}_o} = \frac{\mathcal{E}(T_1)}{\mathcal{E}_o}\, e^{-2\,\mathrm{Im}\,\Omega(T_1)(T-T_1)} \quad , \quad T \geq T_1 \tag{3.121}$$

$$\frac{\mathcal{E}(T_2)}{\mathcal{E}_o} = e^{-2} \tag{3.122}$$

When $T_2 \geq T_1$ we have, from (3.120)-(3.122),

$$e^{-2} = \mathcal{E}(T_2)/\mathcal{E}_o = \left\{\mathcal{E}(T_1)/\mathcal{E}_o\right\} e^{-2\,\mathrm{Im}\,\Omega(T_1)\cdot(T_2-T_1)}$$

$$= \left\{\mathcal{E}(T_1)/\mathcal{E}_o\right\} e^{-2\,\mathrm{Im}\,\Omega(T_1)\cdot T_1\cdot(T_2/T_1 - 1)}$$

$$= \left\{\mathcal{E}(T_1)/\mathcal{E}_o\right\}^{T_2/T_1}$$

Therefore

$$\mathcal{E}(T_1)/\mathcal{E}_o = e^{-2T_1/T_2} \tag{3.123}$$

It follows from (3.123) that if $T_1 = T_2$, only *13.5%* of the original mean field energy is left by the time the dispersion relation stabilizes. We must therefore require that $T_1 << T_2$ if we wish to study the decay of the mean field solely in terms of the stabilized dispersion. For example, if $T_1 \leq 0.05\ T_2$, no more than *10%* of the initial energy will be lost during the time-dependent portion of the decay.

We may study the restriction imposed by the condition $T_1 < T_2$ for the case of Gaussian turbulence in an incompressible fluid by examining the behaviour of solutions of (3.104). Results obtained in this way should be qualitatively valid despite the reservations mentioned above concerning the behaviour of solutions of this equation at large times. In order to simplify the discussion, we shall replace (3.104) with the right hand side of the inequality (3.105), since, as noted above, the results obtained using these two equations are not substantially different over most of the allowed range of $\mathrm{Im}\ \Omega/\eta K^2$.

3.8.5 <u>Conditions on the turbulence for stable decay to be established before significant energy is lost from the mean field</u>

We shall first derive a necessary condition on $(\lambda_c K)^2/q$ for T_1 to be less than T_2 . From either (3.104) or (3.105) we see that T_1 is determined by the requirement that the integrand on the right hand side be effectively zero - i.e.

$$-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\mathrm{Im}\frac{\Omega}{\eta K^2} - \frac{1}{1+2x}\right\}x - \frac{5}{2}\ln(1+2x)$$

$$< -N \quad , \quad \forall\ x > x_1 \qquad (3.124)$$

where $x = T/q\tau_c$, $x_1 = T_1/q\tau_c$, and N is a positive

number for which e^{-N} is fairly small (say N = 2). The requirement $T_1 < T_2$ is satisfied if $x_1 < x_2$. where, by (3.120) and (3.122),

$$x_2 = T_2/q\tau_c = \{q\tau_c \mathcal{I}m\,\Omega\}^{-1} = \{(\lambda_c K)^2 \mathcal{I}m\,\Omega/\eta K^2\}^{-1} \tag{3.125}$$

In the limit $q \to \infty$, (3.124) is satisfied for all $x > x_1 \equiv \sqrt{2N}/q$, so that the condition $x_1 < x_2$ implies that

$$\frac{(\lambda_c K)^2}{q} \mathcal{I}m \frac{\Omega}{\eta K^2} < \frac{1}{\sqrt{2N}} < 1$$

But $\text{Im}\,\Omega/\eta K^2 \geqslant 1$. Hence a necessary condition for $T_1 < T_2$ is

$$\frac{(\lambda_c K)^2}{q} = \eta K^2 \tau_c \ll 1 \tag{3.126}$$

Similarly, when $q \to 0$ (3.124) can only be satisfied if x is sufficiently large for the first two terms on the left hand side to give a *negative* contribution - i.e.

$$x_1 > 2\frac{(\lambda_c K)^2}{q^2} \mathcal{I}m \frac{\Omega}{\eta K^2} \tag{3.127}$$

From (3.125), (3.127), and the condition that $\text{Im}\,\Omega/\eta K^2 \geqslant 1$ (3.126) again gives a necessary condition for $T_1 < T_2$. *(3.126) may therefore be taken as a necessary condition on* $(\lambda_c K)^2/q$ *for all values of* q .

Numerical study of (3.104), or, to a reasonable approximation, of the right hand side of (3.105), leads to more precise conditions on the form of the turbulence for

$T_1 < T_2$. *Figure 8* shows a plot of

$$\left\{R_m'/q\right\}_{T_1=T_2} = \left\{\sqrt{\overline{u^2}}\,\tau_c/\lambda_c\right\}_{T_1=T_2}$$

obtained from (3.105) as a function of λ_c/L for several values of q . At points to the right of each curve, $T_1 > T_2$ and the mean field decay rate is effectively time-dependent throughout the decay. In plotting these curves we have determined T_1 by assuming the time dependence of $\text{Im}\,\Omega(T)$ to be effectively linear during the time-dependent portion of the decay. From (3.104) or (3.105), we see that

$$\left\{\frac{\partial}{\partial t}\,\mathcal{I}m\,\frac{\Omega}{\eta K^2}\right\}_{T=0} = \frac{(R_m')^2}{3q\tau_c}$$

Thus, making use of the initial condition (3.112a) on $\overline{\underset{\sim}{B}}$,

$$\begin{aligned}\mathcal{I}m\,\frac{\Omega}{\eta K^2} &\approx \frac{(\mathcal{I}m\,\Omega)_0}{\eta K^2} + \frac{(R_m')^2}{3q\tau_c}T + \cdots \\ &\approx 1 + \frac{(R_m')^2}{3q}\left(T/\tau_c\right) + \cdots \end{aligned} \qquad (3.128)$$

Using (3.128) we may write, approximately,

$$\left(R_m'/q\right)^2 \approx \frac{3\left\{\mathcal{I}m\,(\Omega/\eta K^2) - 1\right\}}{q\,(T/\tau_c)} \qquad (3.129)$$

The value of T_1 is then determined by equating $\left\{R_m'(T_1)/q\right\}^2$ defined by (3.129), with the value of $\left\{R_m'/q\right\}^2$ obtained from the right hand side of (3.105) in the limit $T \to \infty$, where $T \equiv (t-t_0)$.

$$\left(T_1/\tau_c\right) = q\int_0^\infty e^{-\frac{1}{2}q^2x^2 + q\tau_c\{\mathcal{I}m\,\Omega(T_1) - \frac{\eta K^2}{1+2x}\}x}\,\frac{dx}{(1+2x)^{5/2}} \tag{3.130}$$

From (3.125) the value of T_2 is given by

$$\left(T_2/\tau_c\right) = \{\tau_c\,\mathcal{I}m\,\Omega(T_2)\}^{-1} = \frac{q}{(\lambda_c K)^2\,\mathcal{I}m\,\Omega(T_2)/\eta K^2} \tag{3.131}$$

The condition $T_1 \leqslant T_2$ then implies that

$$\begin{aligned}\left(T_1/\tau_c\right) \leqslant \left(T^*/\tau_c\right) &\equiv \\ &\equiv q\int_0^\infty e^{-\frac{1}{2}q^2x^2 + q\{(\tau_c/T^*) - \frac{(\lambda_c K)^2}{q(1+2x)}\}x}\,\frac{dx}{(1+2x)^{5/2}}\end{aligned} \tag{3.132}$$

Equation (3.132) may be solved numerically or graphically for (T^*/τ_c) .

Equations (3.129), (3.131), and (3.132) lead to the identification

$$\mathcal{I}m\,\Omega(T^*)/\eta K^2 = \frac{q}{(\lambda_c K)^2}\left(T^*/\tau_c\right)^{-1} \tag{3.133}$$

$$\left\{R_m'/q\right\}^2_{T^*} = \frac{3\{q/(\lambda_c K)^2 - (T^*/\tau_c)\}}{q\,(T^*/\tau_c)^2} \tag{3.134}$$

From (3.134) it is clear that $[R_m'/q]_T{}^*$ is real only if

$$\frac{(\lambda_c K)^2}{q} \leqslant \left(T^*/\tau_c\right)^{-1} \tag{3.135}$$

Thus, in a plot of $[R_m'/q]_T{}^*$ against $\lambda_c K$ for a fixed value of q , there will be a critical value of $\lambda_c K$ beyond which no solutions exist. This behaviour is illustrated in *Figure 8*. As q approaches zero, it is clear from (3.126) that $\lambda_c K$ must also go to zero. It follows that when q

is small, it is only appropriate to discuss the decay of the mean field in terms of the *stabilized* decay rate if $\lambda_c K$ is also small. In the limit as $q \to \infty$, the solution of (3.132) gives

$$T_1/\tau_c \leqslant \left(T^*/\tau_c\right)_{q=\infty} = 2.0 \qquad (3.136)$$

so that, from (3.133) and (3.134),

$$\lim_{q\to\infty} \mathcal{I}m\,\Omega(T^*)/\eta K^2 = 2q/(\lambda_c K)^2 = q/\{2\pi^2(\lambda_c/L)^2\} \qquad (3.137)$$

$$\lim_{q\to\infty} \left(R_m'/q\right)^2_{T^*} = 3/4(\lambda_c K)^2 = 3/\{4\pi(\lambda_c/L)\}^2 \qquad (3.138)$$

Solutions of (3.132) are shown in *Figure 9*, where (T^*/τ_c) is plotted as a function of q for several values of λ_c/L. It is clear from (3.132) that the largest values of T^*/τ_c occur in the limit $q \to \infty$, so that, from (3.136), *stabilization* of the dispersion relation must take place in less than two correlation times if T_1 is to be less than T_2. At low values of q, stabilization must take place more rapidly, as indicated in *Figure 9*.

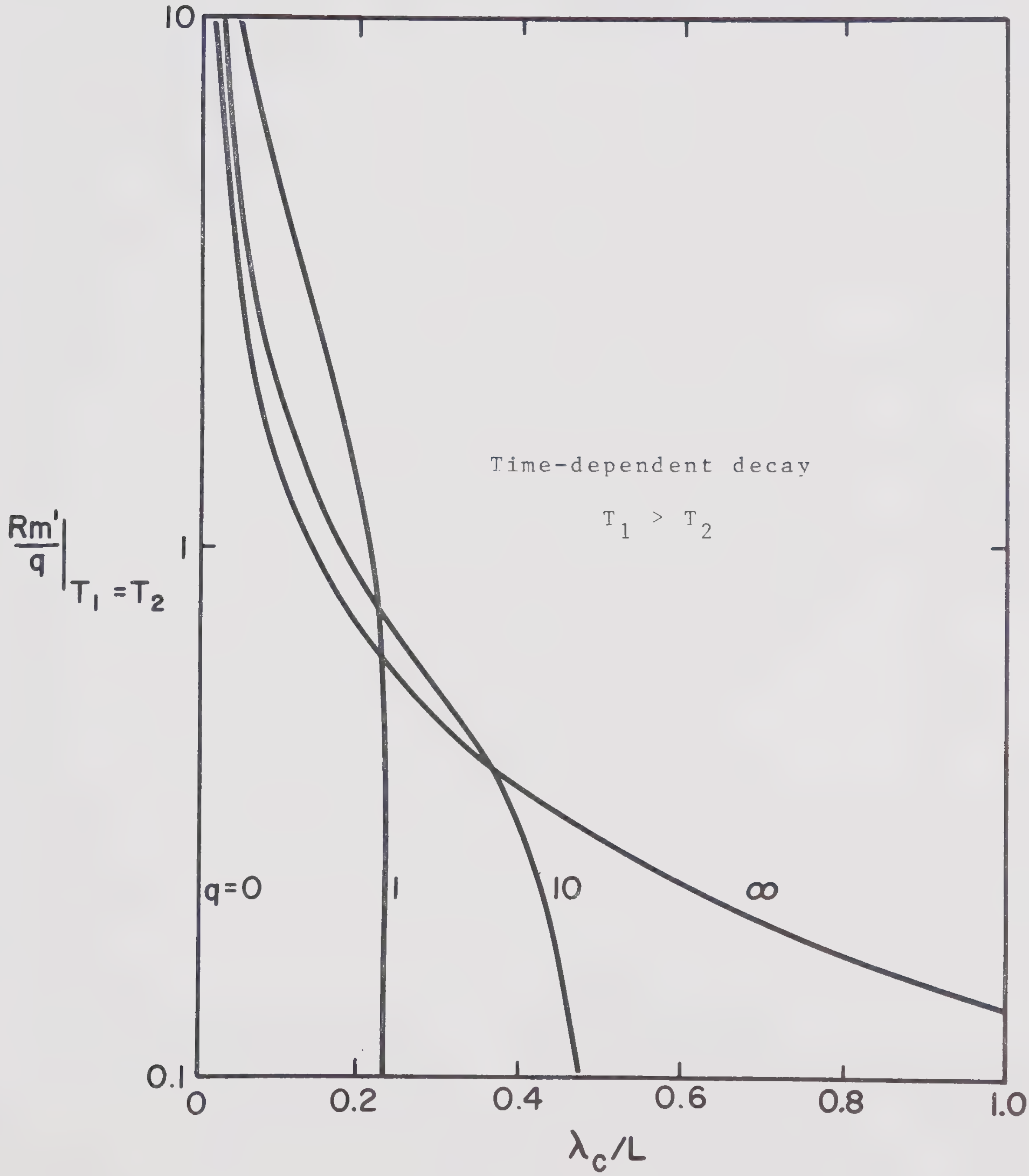
It should be noted that in all plots, λ_c/L is restricted to values less than unity. A situation in which the wavelength of the mean field was shorter than the correlation length of the turbulence (i.e. $\lambda_c/L > 1$) would clearly have little physical significance.

The program used to calculate $T^*/\ _c$ and $[R_m'/q]_{T^*}$ is listed in *Appendix 3, section A.3.2.*

Figure 8. Decay regimes for initial condition *I*.

The plot shows $[R_m'/q]_{T_1=T_2}$ as a function of λ_c/L for several values of q. For each q, decay is effectively time-dependent at all times for points to the *right* of the curve.

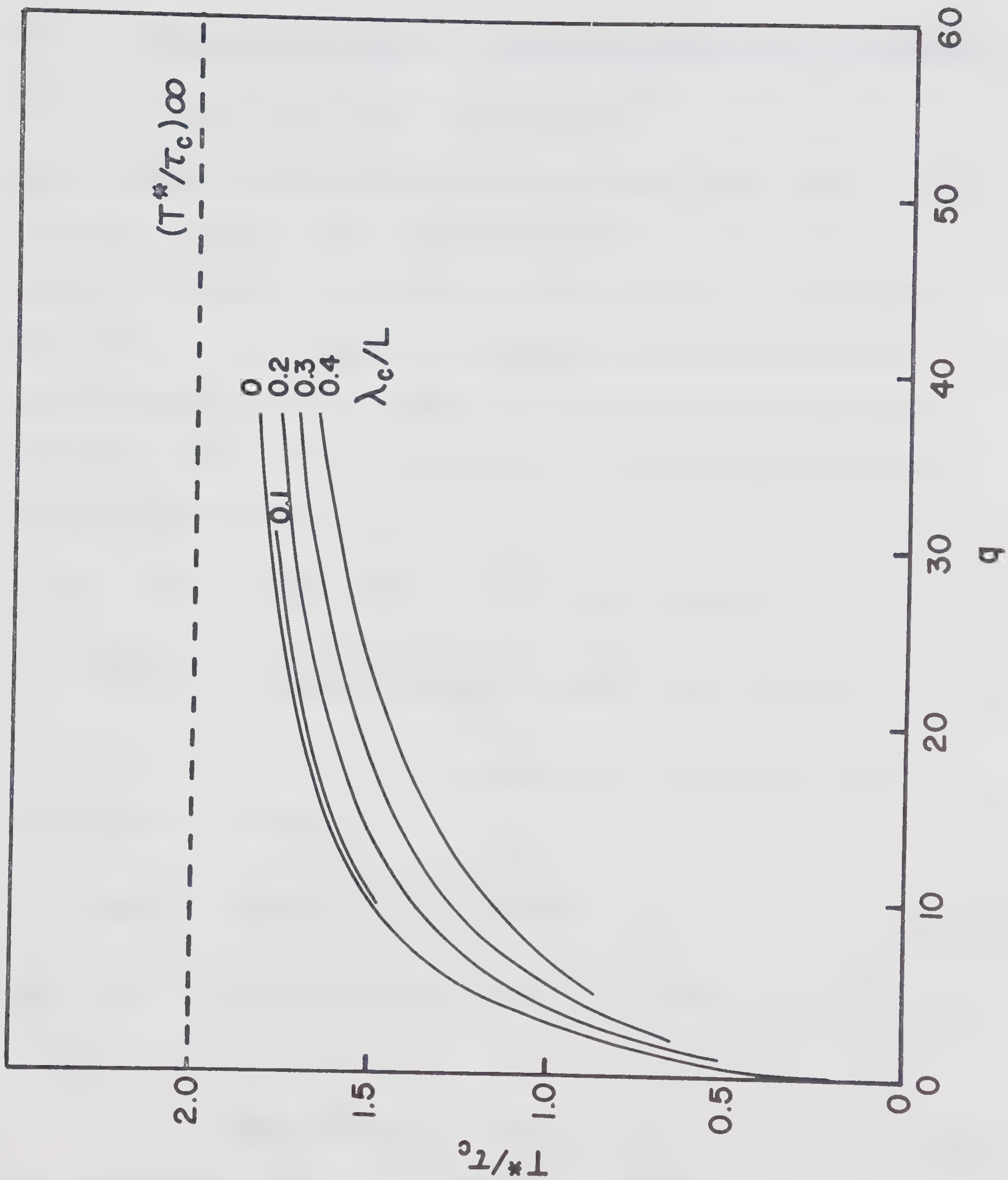
Values plotted are determined from equation (3.134).



Time-dependent decay
$T_1 > T_2$
$\left.\frac{Rm'}{q}\right|_{T_1=T_2}$
10
1
0.1
q=0
1
10
∞
0
0.2
0.4
0.6
0.8
1.0
λ_c/L

Figure 9. T^*/τ_c as a function of q for several values of λ_c/L .

Values plotted represent numerical solutions of equation (3.132).



$(T^*/\tau_c)_\infty$
0
0.1
0.2
0.3
0.4
λ_c/L
T^*/τ_c
0
0.5
1.0
1.5
2.0
0
10
20
30
40
50
60
q

3.9 Isotropic turbulence and decaying mean fields which oscillate with time - initial condition *I*

3.9.1 Inappropriateness of the *Rädler* expansion technique

We shall now turn our attention to the case in which the decaying mean field oscillates with time - i.e. $\mathrm{Re}\,\Omega \neq 0$. It is not immediately obvious that either (3.35) or (3.39) has solutions of this type. The Rädler expansion (3.96), which corresponds to (3.35) is of no use in studying the problem, as may be seen by taking the first few terms of the expansion and separating the real and imaginary parts.

$$\mathcal{I}m\,\Omega - \eta K^2 = \overline{u^2}K^2\{\beta^{(00)} - \beta^{(10)}K^2 + \beta^{(01)}\mathcal{I}m\,\Omega + \dots\} \tag{3.139a}$$

$$\mathcal{R}e\,\Omega = \mathcal{R}e\,\Omega\{\overline{u^2}K^2\}\{\beta^{(01)} - \beta^{(11)}K^2 + \beta^{(02)}\mathcal{I}m\,\Omega + \dots\} \tag{3.139b}$$

If $\mathrm{Re}\,\Omega = 0$, (3.139b) is identically satisfied, and (3.139a) is a meaningful equation of the form

$$\mathcal{O}(\epsilon) = \mathcal{O}(\epsilon)\cdot\overline{u^2}\,\{1 + \mathcal{O}(\epsilon) + \dots\} \tag{3.140a}$$

where ϵ is assumed to be small. However, when $\mathrm{Re}\,\Omega \neq 0$ (3.139b) is of the form

$$1 = \mathcal{O}(\epsilon)\cdot\overline{u^2}\,\{1 + \mathcal{O}(\epsilon) + \dots\} \tag{3.140b}$$

and is clearly incompatible with (3.140a) under the assumption that ϵ is small. We must therefore study the problem in terms of the unexpanded equations, taking note

of their *eigenvalue* nature, discussed above in section 3.4.3.

3.9.2 General statement of the problem

Consider first equation (3.35), associated with *initial condition I* on $\underset{\sim}{B}'$. For the reasons set out above in section 3.7.3, we shall restrict ourselves to the case of Gaussian turbulence (3.67), for which (3.35) reduces to (3.103). Separating the real and imaginary parts of (3.103),

$$\mathcal{I}m\{\Omega(T)/\eta K^2\} - 1 = (R_m')^2 \, J_R(q, \lambda_c K\, ; T/\tau_c) \qquad (3.141a)$$

$$\mathcal{R}e\{\Omega(T)/\eta K^2\} = -(R_m')^2 \, J_I(q, \lambda_c K\, ; T/\tau_c) \qquad (3.141b)$$

where

$$J_R(q, \lambda_c K; T/\tau_c) \equiv$$

$$\equiv \frac{1}{3}\int_0^{T/q\tau_c} e^{-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\mathcal{I}m\frac{\Omega}{\eta K^2} - \frac{1}{1+2x}\right\}x} \cdot$$

$$\cdot \cos\left\{(\lambda_c K)^2\, \mathcal{R}e\frac{\Omega}{\eta K^2}x\right\} \frac{dx}{(1+2x)^{5/2}} \qquad (3.142a)$$

$$J_I(q, \lambda_c K\, ; T/\tau_c) \equiv$$

$$\equiv \frac{1}{3}\int_0^{T/q\tau_c} e^{-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\mathcal{I}m\frac{\Omega}{\eta K^2} - \frac{1}{(1+2x)}\right\}x} \cdot$$

$$\cdot \sin\left\{(\lambda_c K)^2\, \mathcal{R}e\frac{\Omega}{\eta K^2}x\right\} \frac{dx}{(1+2x)^{5/2}} \qquad (3.142b)$$

Once again, it must be stressed that (3.142a,b) are *integral equations*. To a first approximation, they may be replaced by

$$\mathcal{I}m\{\Omega(T)/\eta K^2\} - 1 = (R_m')^2 J_{2R}(q, \lambda_c K; T/\tau_c) \tag{3.143a}$$

$$\mathcal{R}e\{\Omega(T)/\eta K^2\} = -(R_m')^2 J_{2I}(q, \lambda_c K; T/\tau_c) \tag{3.143b}$$

where

$$J_{2R}(q, \lambda_c K; T/\tau_c) \equiv$$
$$\equiv \frac{1}{3}\int_0^{T/q\tau_c} e^{-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\mathcal{I}m\frac{\Omega(T)}{\eta K^2} - \frac{1}{1+2x}\right\}x} \cdot \cos\left\{(\lambda_c K)^2\,\mathcal{R}e\,\frac{\Omega(T)}{\eta K^2}\,x\right\}\frac{dx}{(1+2x)^{5/2}} \tag{3.144a}$$

$$J_{2I}(q, \lambda_c K; T/\tau_c) \equiv$$
$$\equiv \frac{1}{3}\int_0^{T/q\tau_c} e^{-\frac{1}{2}q^2x^2 + (\lambda_c K)^2\left\{\mathcal{I}m\frac{\Omega(T)}{\eta K^2} - \frac{1}{1+2x}\right\}x} \cdot \sin\left\{(\lambda_c K)^2\,\mathcal{R}e\,\frac{\Omega(T)}{\eta K^2}\,x\right\}\frac{dx}{(1+2x)^{5/2}} \tag{3.144b}$$

and solutions may be sought in the limit as $T \to \infty$. As discussed above in section 3.8.4, the parameters of the turbulence must be chosen in such a way that $T_1 << T_2$ if the solutions of (3.143a,b) are to have any physical significance.

Dr. K.D. Aldridge and the present author have developed a numerical program which evaluates the integrals

J_{2R} and J_{2I} by a *Simpson's Rule* technique, and calculates the dispersion function

$$D \equiv 1 - \frac{\mathcal{R}e\{\Omega/\eta K^2\}}{\mathcal{I}m\{\Omega/\eta K^2\} - 1} \cdot \frac{J_{2R}(q, \lambda_c K; \infty)}{J_{2I}(q, \lambda_c K; \infty)} \qquad (3.145)$$

D must vanish for a solution of (3.143a,b) with $\mathrm{Re}\,\Omega \neq 0$ to exist. When this condition is satisfied, the eigenvalue for the problem may be calculated from

$$(R_m')^2 = -\frac{\mathcal{R}e\{\Omega/\eta K^2\}}{J_{2I}(q, \lambda_c K; \infty)} \qquad (3.146)$$

3.9.3 Restrictions on the turbulence for meaningful solutions

In order for the problem to have meaningful solutions, the parameters of the turbulence must satisfy a number of conditions, which are summarized here for convenience.

$$\lambda_c/L < 1 \quad , \quad \tau_c/T < 1 \tag{3.147a,b}$$

$$\tau_c \eta K^2 = (\lambda_c K)^2/q \ll 1 \tag{3.148}$$

$$\{R_m'/q\} < \{R_m'/q\}_{T*}(q, \lambda_c/L) \tag{3.149a}$$

$$\{\lambda_c/L\} < \{\lambda_c/L\}_{T*}(q, R_m'/q) \tag{3.149b}$$

$$R_m' < 1 + q \tag{3.150}$$

In (3.147), L and T are the wavelength and period of the mean field $\bar{\underset{\sim}{B}}$. (3.148) and (3.149) are the conditions derived in section 3.8.5 for T_1 T_2, and the functions referred to in (3.149) are plotted in *Figure 8*. (3.150) is the condition derived in section 2.5.4 for the first order smoothing approximation to be consistent.

First order smoothing also implies that

$$\mathrm{Im}\,\Omega/\eta K^2 \gg 1 \tag{3.151}$$

when the turbulence is PT-invariant, as proved in section 3.2. This condition may be combined with (3.147)-(3.150)

to derive further restrictions on the parameters of the mean field.

From (3.147b) and (3.148) we have

$$2\pi\,\tau_c/T \ll \mathcal{R}e\,\Omega/\eta K^2 < 2\pi\, q/(\lambda_c K)^2 \qquad (3.152)$$

or, writing $T_d = 1/\eta K^2$,

$$\tau_c \ll \sqrt{T \cdot T_d} \qquad (3.152')$$

Similarly, from (3.151) we have

$$T_2/\tau_c < q/(\lambda_c K)^2 \qquad (3.153)$$

The effective range of the arguments of the sine and cosine terms in (3.144a,b) is limited by

$$\begin{aligned}(\lambda_c K)^2\, \mathcal{R}e\{\Omega/\eta K^2\}\, x &\le (\lambda_c K)^2\, \mathcal{R}e\{\Omega/\eta K^2\}\, x_1 \\ &< 2\pi\, T_1/\tau_c \\ &< 2\pi\{T^*/\tau_c\}(q, \lambda_c K) \\ &< 4\pi \end{aligned} \qquad (3.154)$$

where (T^*/τ_c) is the function plotted in *Figure 9*.

3.9.4 Numerical search procedure

A numerical search has failed to reveal *any* long-period oscillatory solutions of (3.143a,b) which satisfy (3.147) and (3.148). Asymptotic evaluation of the integrals J_{2R} and J_{2I} indicates that oscillatory solutions do exist when $(\lambda_c K)^2 >> 1$, but the program used was unable to check the existence of solutions in this range because of the extremely large values attained by the exponential factors in the integrands in (3.143).

The procedure used in the numerical search was briefly as follows.

a) Trial values of $\mathrm{Im}\ \Omega/\eta K^2$ and $\mathrm{Re}\ \Omega/\eta K^2$ were chosen.

b) A value of q was assumed.

c) The integrals J_{2R} and J_{2I}, and the dispersion function D were calculated for a range of values of $\lambda_c K$ satisfying (3.147a), (3.148), and (3.152).

d) The calculated values of D were checked for changes of sign or trends toward zero.

e) Further values of $\lambda_c K$ were chosen in accordance with the results of step (d), until the conditions (3.147a), (3.148), and (3.152) could no longer be satisfied.

f) The process was repeated for new values of q, and for new trial values of $\mathrm{Im}\ \Omega/\eta K^2$ and $\mathrm{Re}\ \Omega/\eta K^2$.

As the trial values of $\mathrm{Re}\ \Omega/\eta K^2$ chosen were kept fairly small (in general, much less than unity), the entire range of $\mathrm{Re}\ \Omega/\eta K^2$ permitted by (3.152) was not explored at large values of q. It is therefore possible that acceptable oscillatory solutions to (3.143) do exist when both q and $\mathrm{Re}\ \Omega/\eta K^2$ are large. The assumption of small $\mathrm{Re}\ \Omega/\eta K^2$, which implies that the period sought is long compared with the mean field decay time, was made in order to restrict consideration to small departures from the case $\mathrm{Re}\ \Omega = 0$.

See *Appendix 3, section A.3.3* for details of the numerical techniques used in evaluating the integrals J_{2R} and J_{2I}.

3.9.5 The impossibility of slowly-decaying, long-period, oscillatory mean fields

It may be demonstrated directly that (3.143) has no oscillatory solutions when

$$\frac{(\lambda_c K)^2}{q} \left| 1 + i \frac{\Omega}{\eta K^2} \right| \ll 1$$

The equations (3.143) may be rewritten in the form

$$\Gamma = \frac{(R_m')^2}{3q} \int_0^\infty e^{-\frac{1}{2}y^2 + \gamma \Gamma y + \frac{2\gamma y^2}{q+2y}} \frac{dy}{(1+2y/q)^{5/2}} \tag{3.155}$$

where

$$\Gamma \equiv -\left\{1 + i\frac{\Omega}{\eta K^2}\right\} = \left\{\mathrm{Im}\frac{\Omega}{\eta K^2} - 1\right\} - i\,\mathrm{Re}\frac{\Omega}{\eta K^2} \tag{3.156}$$

$$\gamma \equiv (\lambda_c K)^2/q \tag{3.157}$$

The integrand in (3.155) may be expanded as a Taylor series in $\gamma\Gamma$ about $\gamma\Gamma = 0$, and integrated term by term to give

$$\Gamma = \frac{(R_m')^2}{3q} \sum_{n=0}^{\infty} \frac{1}{n!} a_n (\gamma\Gamma)^n \tag{3.158}$$

where

$$a_n \equiv \int_0^\infty y^n e^{-\frac{1}{2}y^2 + \frac{2\gamma y^2}{q+2y}} \frac{dy}{(1+2y/q)^{5/2}} \tag{3.159}$$

The series in (3.158) converges absolutely for all values of $\gamma\Gamma$ by *d'Alembert's* ratio test *(Whittaker and Watson, 1927, p. 22)*.

Equation (3.158) will clearly have no complex solutions for Γ as long as $|\gamma\Gamma|$ is small enough for the quadratic term on the right hand side to be ignored. This condition may be written

$$|\gamma\Gamma| \ll 2a_1/a_2 \qquad (3.161)$$

When q is large, (3.161) may be replaced by

$$(q \to \infty) \qquad |\gamma\Gamma| \ll 2\sqrt{2/\pi} \qquad (3.161')$$

and when q is small, (3.161) becomes approximately

$$(q \to 0) \qquad |\gamma\Gamma| \ll 2/(3\sqrt{q}) \qquad (3.161'')$$

It would appear, therefore, that complex solutions will be found most readily when q is large. In this limit,

$$\lim_{q\to\infty} a_{2n} = \int_0^\infty y^{2n} e^{-\frac{1}{2}y^2} dy = \frac{(2n)!}{2^n n!}\sqrt{\frac{\pi}{2}}$$

$$\lim_{q\to\infty} a_{2n+1} = \int_0^\infty y^{2n+1} e^{-\frac{1}{2}y^2} dy = 2^n n!$$

When the quadratic term in (3.158) is retained, the solution is

$$\Gamma = \frac{1}{\gamma a_2}\left\{\frac{1}{\gamma(R_m')^2} - a_1 \pm \sqrt{\left[\frac{3q}{\gamma(R_m')^2} - a_1\right]^2 - 2a_0a_2}\right\} \qquad (3.162)$$

Γ will be complex if

$$2a_0a_2 > \left\{\frac{3q}{\gamma(R_m')^2} - a_1\right\}^2 \qquad (3.163)$$

or, taking the limit of large q ,

$$(q \to \infty) \qquad \left\{\frac{(\lambda_c K) R_m'}{q}\right\}^2 > \frac{3}{1+\sqrt{\pi}} \tag{3.163'}$$

(3.163') may be compared with the condition (3.138), which must be satisfied if T_1 is to be less than T_2.

$$(q \to \infty) \qquad \left\{\frac{(\lambda_c K) R_m'}{q}\right\}^2 < \frac{3}{4} \tag{3.138'}$$

Clearly, the two conditions are incompatible, indicating that no acceptable oscillatory solutions to (3.143) are to be found when $|\gamma\Gamma|$ is small enough for the cubic term in (3.158) to be neglected - i.e. when

$$|\gamma\Gamma| << 3a_2/a_3$$

In the limit of large q , this condition becomes

$$(q \to \infty) \qquad |\gamma\Gamma| << 3\sqrt{\pi}/2\sqrt{2} \simeq 2$$

From (3.156) and (3.157),

$$|\gamma\Gamma| = \gamma|\Gamma| = \frac{(\lambda_c K)^2}{q}\left\{\left(\mathcal{I}m\frac{\Omega}{\eta K^2} - 1\right)^2 + \left(\mathcal{R}e\frac{\Omega}{\eta K^2}\right)\right\}^{1/2}$$

Thus $|\gamma\Gamma| << 2$ only if

$$\frac{(\lambda_c K)^2}{q}\mathcal{R}e\frac{\Omega}{\eta K^2} = \tau_c \mathcal{R}e\,\Omega \ll 2$$

and

$$\frac{(\lambda_c K)^2}{q}\left\{\mathcal{I}m\frac{\Omega}{\eta K^2} - 1\right\} = \tau_c\left\{\mathcal{I}m\,\Omega - \eta K^2\right\} \ll 2$$

But, by (3.148),

$$\gamma = \eta \tau_c K^2 << 1$$

It follows that $|\gamma\Gamma| << 2$ only if the period of the mean field

$$T = 2\pi/\mathrm{Re}\,\Omega$$

and the effective decay time

$$T_d^{eff} \equiv 1/\mathrm{Im}\,\Omega$$

satisfy the conditions

$$\tau_c/T << 1/\pi \quad , \quad \tau_c/T_d^{eff} << 2$$

We may therefore state that (3.143) has no acceptable oscillatory solutions for which both the period and the effective decay time of the mean field are long compared with the correlation time of the turbulence.

By (3.147b), $\tau_c/T < 1$. In the limit as $q \to \infty$ we must also require that T_d^{eff} satisfy

$$T_d^{eff} \equiv T_2 > T_1 \sim 2\tau_c$$

- i.e. when q is large, τ_c/T_d^{eff} is typically less than $\frac{1}{2}$. It follows therefore, that the only range in which acceptable solutions to (3.143) may possibly occur is that in which

$$(q \to \infty) \qquad (1/\pi) \lesssim \tau_c/T < 1$$

This condition confirms the numerical result that the only range in which solutions may possibly lie is that for which both q and $\mathrm{Re}\,\Omega/\eta K^2$ are relatively "large".

3.10 Isotropic turbulence and decaying mean fields which oscillate with time - initial condition *II*.

3.10.1 General statement of the problem, and restrictions on the turbulence for meaningful solutions

We shall now consider equation (3.39), associated with *initial condition II* on $\underset{\sim}{B}'$. For Gaussian turbulence, (3.67), equation (3.39) reduces to

$$-\left\{1+i\frac{\Omega}{\eta K^2}\right\} = \frac{1}{\eta K^2} I^{(1)}(K,\Omega) =$$

$$= \frac{(\lambda_c K)^5}{24\pi}\,\frac{(R_m')^2}{q}\int_{-\infty}^{\infty}d\nu\int_0^{\infty}d\xi\; e^{-\frac{1}{2}\left\{(\lambda_c K\xi)^2+(\eta K^2\tau_c)^2(\nu+\mathrm{Re}\frac{\Omega}{\eta K^2})^2\right\}}\,\cdot$$

$$\cdot\,\Theta\left(\xi,\nu;\mathrm{Im}\frac{\Omega}{\eta K^2}\right) \qquad (3.164)$$

Separating the real and imaginary parts of (3.164),

$$\left\{\mathrm{Im}\frac{\Omega}{\eta K^2}-1\right\} = \frac{1}{\eta K^2}\,\mathrm{Re}\, I^{(1)}(K,\Omega) =$$

$$= \frac{(\lambda_c K)^5}{24\pi}\,\frac{(R_m')^2}{q}\int_{-\infty}^{\infty}d\nu\int_0^{\infty}d\xi\; e^{-\frac{1}{2}\left\{(\lambda_c K\xi)^2+(\eta K^2\tau_c)^2(\nu+\mathrm{Re}\frac{\Omega}{\eta K^2})^2\right\}}$$

$$\cdot\,\mathrm{Re}\,\Theta\left(\xi,\nu;\mathrm{Im}\frac{\Omega}{\eta K^2}\right) \qquad (3.165a)$$

$$\left\{\mathrm{Re}\frac{\Omega}{\eta K^2}\right\} = -\frac{1}{\eta K^2}\,\mathrm{Im}\, I^{(1)}(K,\Omega) =$$

$$= -\frac{(\lambda_c K)^5}{24\pi}\,\frac{(R_m')^2}{q}\int_{-\infty}^{\infty}d\nu\int_0^{\infty}d\xi\; e^{-\frac{1}{2}\left\{(\lambda_c K\xi)^2+(\eta K^2\tau_c)^2(\nu+\mathrm{Re}\frac{\Omega}{\eta K^2})^2\right\}}$$

$$\cdot\,\mathrm{Im}\,\Theta\left(\xi,\nu;\mathrm{Im}\frac{\Omega}{\eta K^2}\right) \qquad (3.165b)$$

where $\mathrm{Re}\,\Theta$ and $\mathrm{Im}\,\Theta$ are defined in (3.42) and (3.43).

In order for the problem to have meaningful solutions, the parameters of the turbulence and the mean field must satisfy a number of conditions, as was the case for solutions corresponding to *initial condition I (see section 3.9.3)*. The conditions (3.147), (3.150), (3.151), and the right hand side of (3.152) apply to solutions of (3.165), as they did to solutions of (3.143). However, because of the assumption that $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ are correlated initially, solutions to (3.165) need not necessarily satisfy the conditions (3.148) and (3.149).

Dr. K.D. Aldridge and the present author have developed a numerical program which evaluates the two-dimensional integrals in (3.165) by means of an *n-point Gaussian scheme*, and calculates the dispersion function

$$D = 1 + \frac{\mathcal{R}e(\Omega/\eta K^2)}{\{\mathcal{I}m(\Omega/\eta K^2) - 1\}} \quad \frac{\mathcal{R}e\, I^{(1)}}{\mathcal{I}m\, I^{(1)}} \tag{3.166}$$

As for the dispersion function defined in (3.145), D must vanish for a solution of (3.165) with $\mathrm{Re}\,\Omega \neq 0$ to exist. When this condition is satisfied, the eigenvalues for the problem may be calculated from

$$\frac{(\lambda_c K)^5}{24\pi} \frac{(R_m')^2}{q} = \tag{3.167}$$

$$= \frac{-\mathcal{R}e(\Omega/\eta K^2)}{\int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\xi \; e^{-\frac{1}{2}\{(\lambda_c K \xi)^2 + (\eta K^2 t_c)^2(\nu + \mathcal{R}e\frac{\Omega}{\eta K^2})^2\}} \, \mathcal{I}m\, \Theta(\xi, \nu; \mathcal{I}m \frac{\Omega}{\eta K^2})}$$

3.10.2 Numerical search procedure

Because of the less stringent conditions on the parameters of the turbulence for *initial condition II*, numerical solutions to (3.165) can be obtained relatively easily. The numerical search procedure used is outlined below.

a) Trial values of $\mathrm{Im}\ \Omega/\eta K^2$ and $\mathrm{Re}\ \Omega/\eta K^2$ were chosen.

b) A value of $(\lambda_c K)^2/q$ was assumed.

c) The integrals in (3.165) and the dispersion function D were calculated for a range of values of $\lambda_c K$ satisfying the condition (3.147a).

d) The calculated values of D were examined for changes of sign, and further values of $\lambda_c K$ were used if necessary to locate a sign change.

e) An iterative interpolation technique was used to determine the values of $\lambda_c K$ and R_m' corresponding to the solution point.

f) The process was repeated for new values of $(\lambda_c K)^2/q$ and for new trial values of $\mathrm{Im}\ \Omega/\eta K^2$ and $\mathrm{Re}\ \Omega/\eta K^2$.

Solutions were obtained for both large and small values of the parameter $|\Gamma|$, defined in (3.156).

The numerical integration was carried out with a fair degree of precision. Typical examples of the functions integrated are shown in *Figures 10-14*. The surfaces plotted correspond to $\mathrm{Im}\ \Omega/\eta K^2 = 1.025$ and $\mathrm{Re}\ \Omega/\eta K^2 = 0.001$ for several different values of $(\lambda_c K)^2/q$ marked by dots on the appropriate solution curve in *Figure 16*. The functions involving $\mathrm{Re}\ \Theta$ (*Figures 10a, 11a, 12a, 13a, and 14a*) were integrated in three sections:

a) $\xi < \sqrt{(\mathrm{Im}\ \Omega/\eta K^2)-1}$, for which $\mathrm{Re}\ \Theta < 0$

b) $\sqrt{(\mathrm{Im}\ \Omega/\eta K^2)-1} < \xi < \xi^*$, where ξ^* was chosen in such a way that the integration spanned the first peak in the integrand, but stopped short of the second peak (if one existed)

c) $\xi > \xi^*$, spanning the second peak.

On the other hand, the functions involving $\mathrm{Im}\ \Theta$ *(Figures 10b, 11b, 12b, 13b, and 14b)* were integrated in a straightforward manner, since these functions have only a single peak. In each integration, limiting values of ξ and ν , if not already specified by "section" boundaries, were chosen by requiring the absolute value of the integrand to fall below a specified fraction of the value at or near the peak.

The integration scheme has been tested for convergence with respect to both the choice of the specified fraction defining the integration limits, and the number

of Gaussian points used. Typical results of the second test are shown in *Figure 15*. In view of these results, most of the integrations were carried out with an 8x8 grid for each of the three integrations involving Re Θ , and a 12x12 or 16x16 grid for the single integration involving Im Θ . A further check on the accuracy of the integration was provided by preliminary calculations in which the ξ and ν integrations were carried out using a *10-point Laguerre* and a *10-point Hermite polynomial technique*, respectively *(see, for example, Abramowitz and Stegun, 1964, §25.4.45 and §25.4.46)*. The results of the Hermite-Laguerre integration were in good agreement with those obtained later using the Gaussian scheme *(see, for example, Abramowitz and Stegun, 1964, §25.4.30)*.

Details of the program used for the Gaussian integration scheme are given in *Appendix 4*.

Figures 10-14. Integrands of (3.165a) and (3.165b) for several values of λ_c/L and τ_c/T .

In each figure, plot (a) shows

$$\mathcal{R}e\,\Theta\{\xi,\nu;\mathcal{I}m\,\Omega/\eta K^2\}\,e^{-\frac{1}{2}\{(\lambda_c K\xi)^2+(\eta K^2\tau_c)^2(\nu+\mathcal{R}e\,\Omega/\eta K^2)^2\}}$$

while plot (b) shows

$$\mathcal{I}m\,\Theta\{\xi,\nu;\mathcal{I}m\,\Omega/\eta K^2\}\,e^{-\frac{1}{2}\{(\lambda_c K\xi)^2+(\eta K^2\tau_c)^2(\nu+\mathcal{R}e\,\Omega/\eta K^2)^2\}}$$

These quantities are plotted as functions of ξ and ν for fixed values of $\mathrm{Im}\,\Omega/\eta K^2$, $\mathrm{Re}\,\Omega/\eta K^2$, $\lambda_c K$, and $\eta K^2\tau_c$. In the figures,

$$\lambda_c/L \equiv \lambda_c K/2\pi$$

$$\tau_c/T \equiv \eta K^2\tau_c/2\pi$$

$$\Omega_d \equiv \eta K^2$$

Figure 10: $\lambda_c/L = 1.6$ $\quad\tau_c/T = 3.5 \times 10^{-4}$

Figure 11: $\lambda_c/L = 0.45$ $\quad\tau_c/T = 3.5 \times 10^{-3}$

Figure 12: $\lambda_c/L = 0.11$ $\quad\tau_c/T = 3.5 \times 10^{-2}$

Figure 13: $\lambda_c/L = 0.07$ $\quad\tau_c/T = 0.1$

Figure 14: $\lambda_c/L = 0.053$ $\quad\tau_c/T = 1.0$

These values are taken from the fourth curve in *Figure 16*. In each case,

$$\mathrm{Im}\,\Omega/\Omega_d = 1.025 \qquad |\mathrm{Re}\,\Omega/\Omega_d| = 0.001$$

Figure 10a. Integrand of (3.165a)

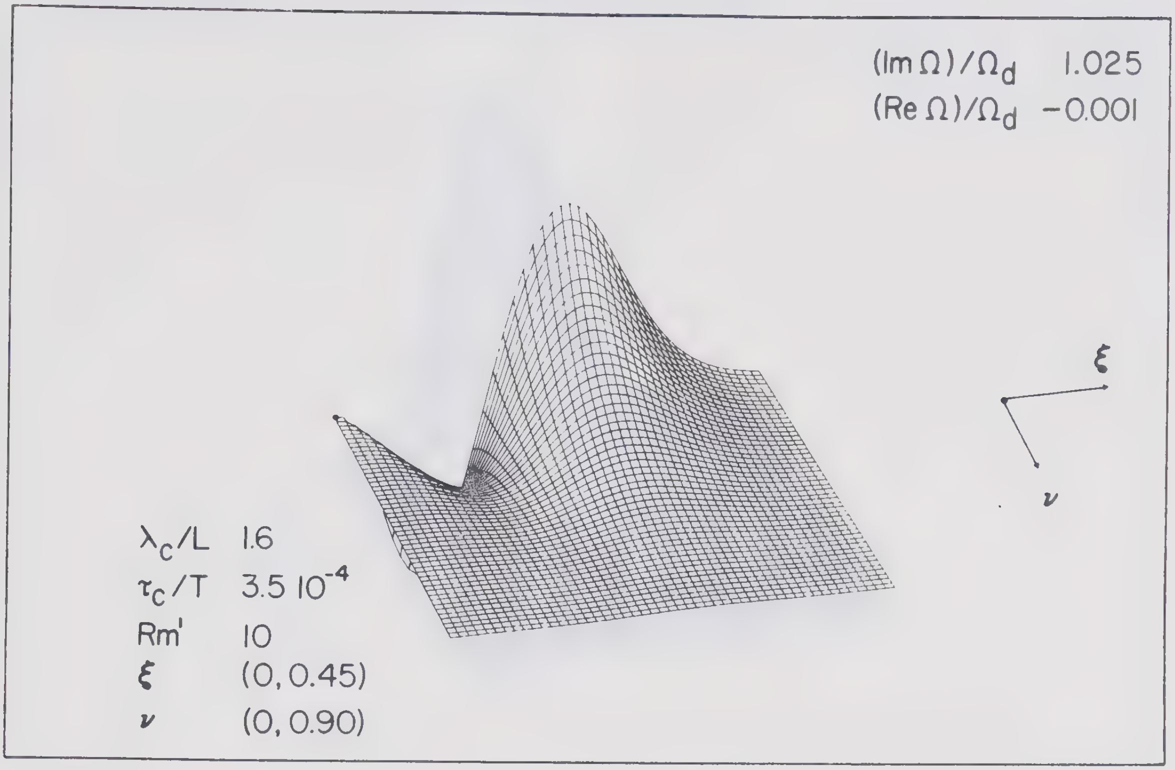


Figure 10b. Integrand of (3.165b)

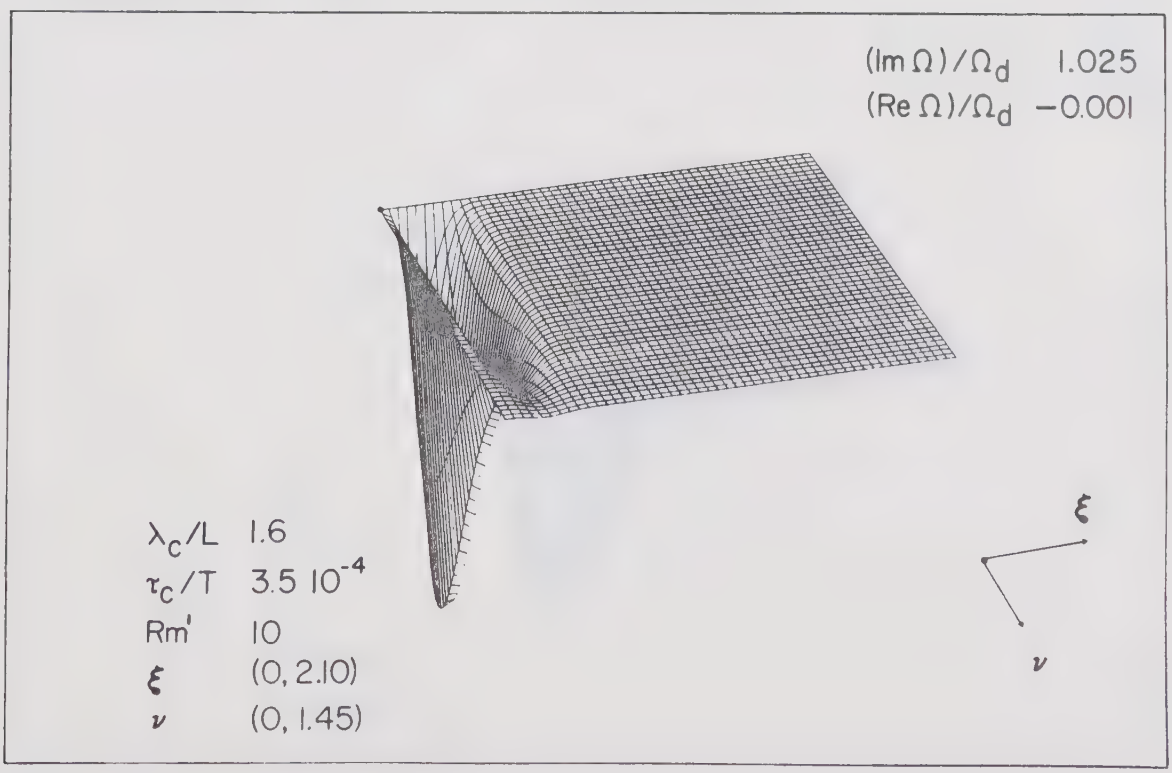


Figure 11a. Integrand of (3.165a)

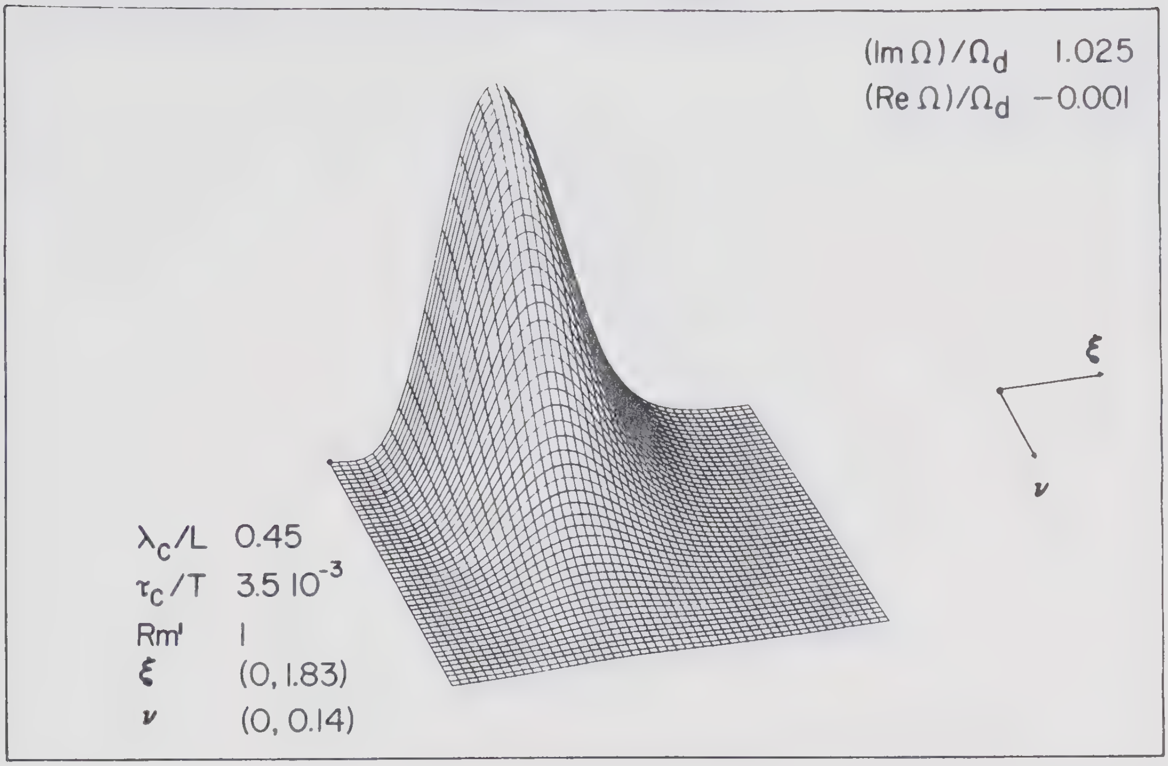


Figure 11b. Integrand of (3.165b)

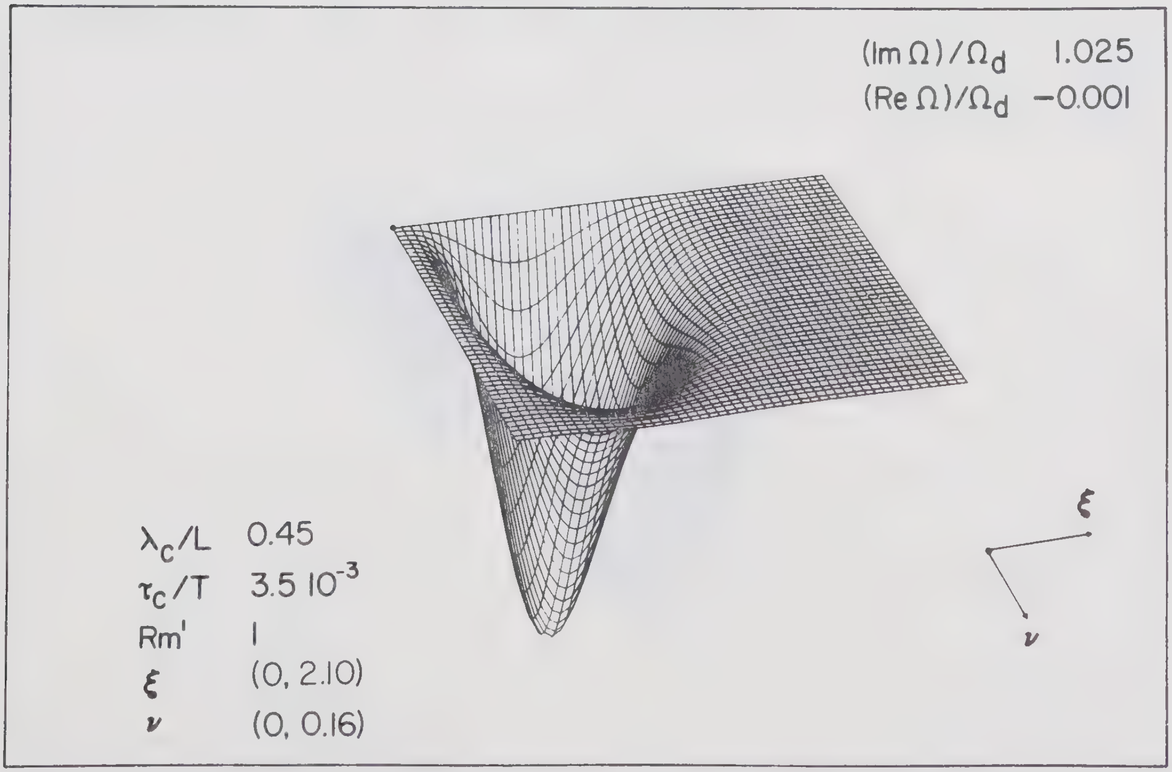


Figure 12a. Integrand of (3.165a)

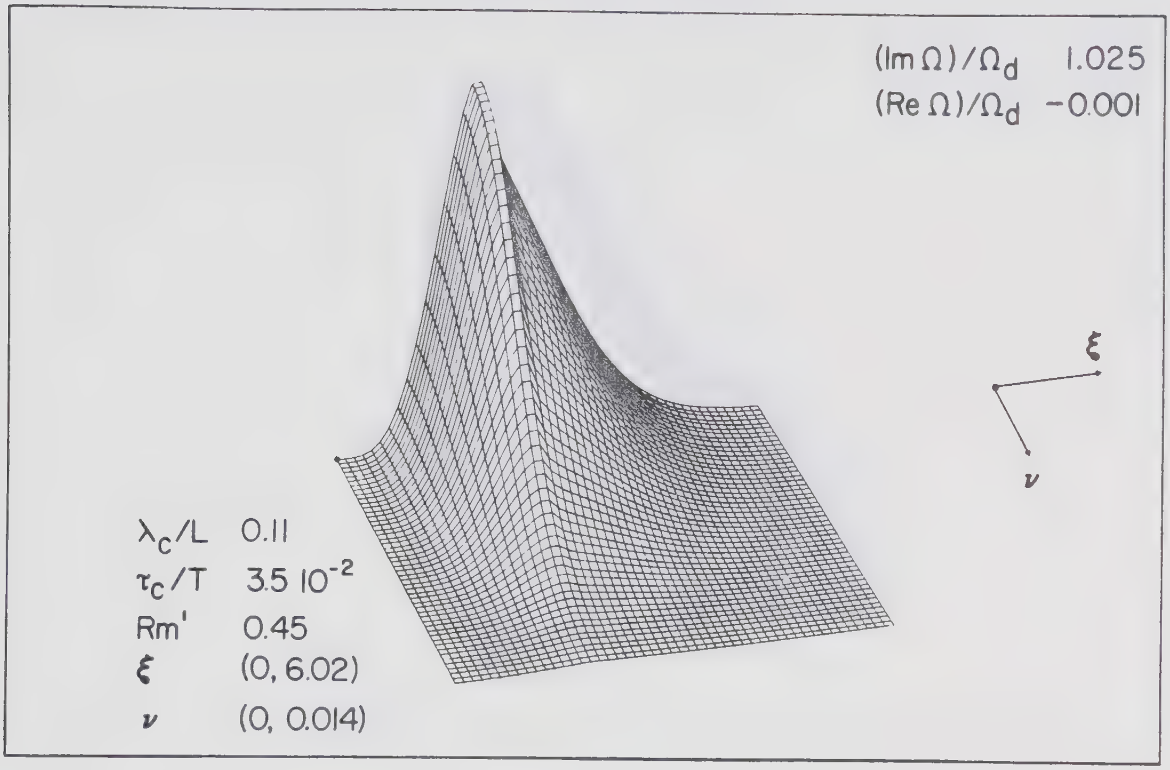


Figure 12b. Integrand of (3.165b)

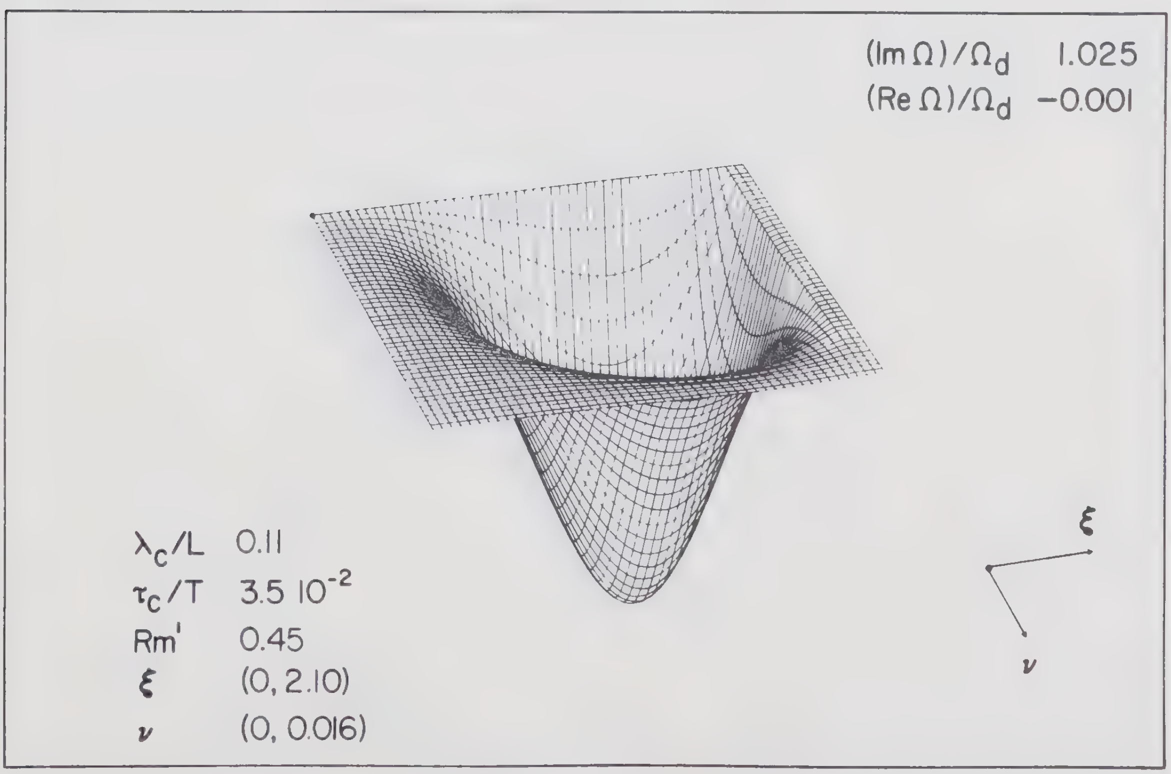


Figure 13a. Integrand of (3.165a)

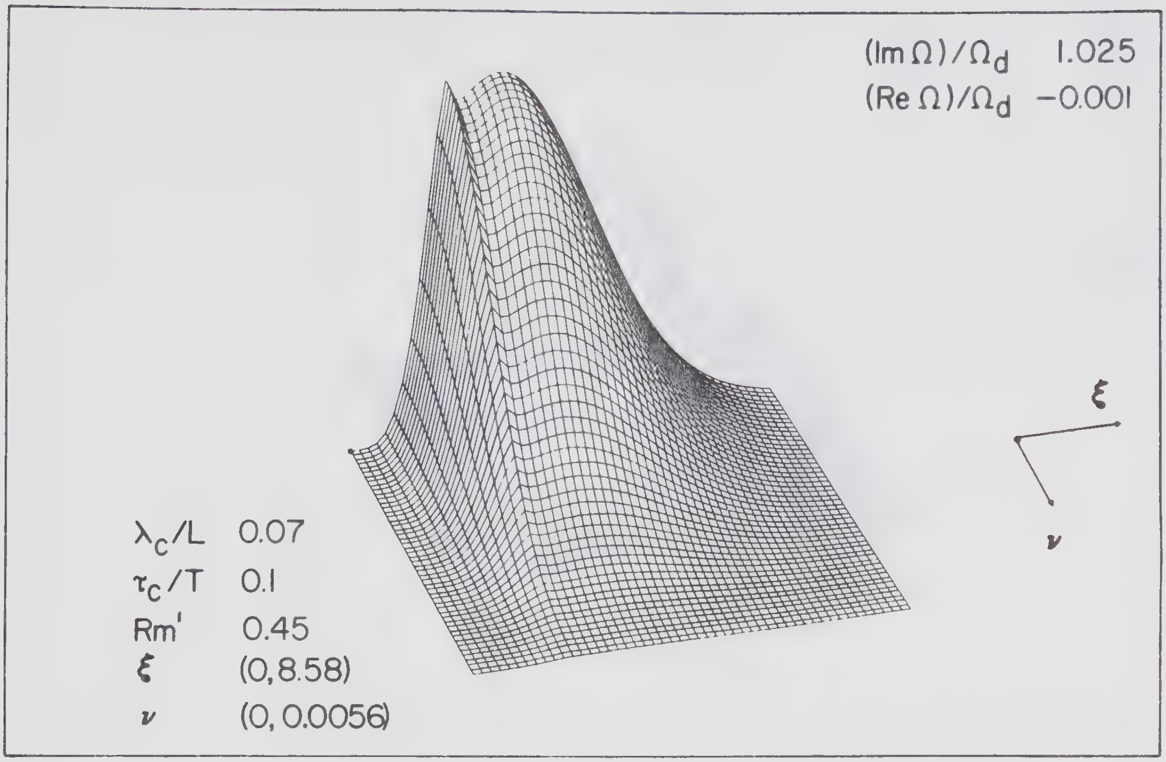


Figure 13b. Integrand of (3.165b)

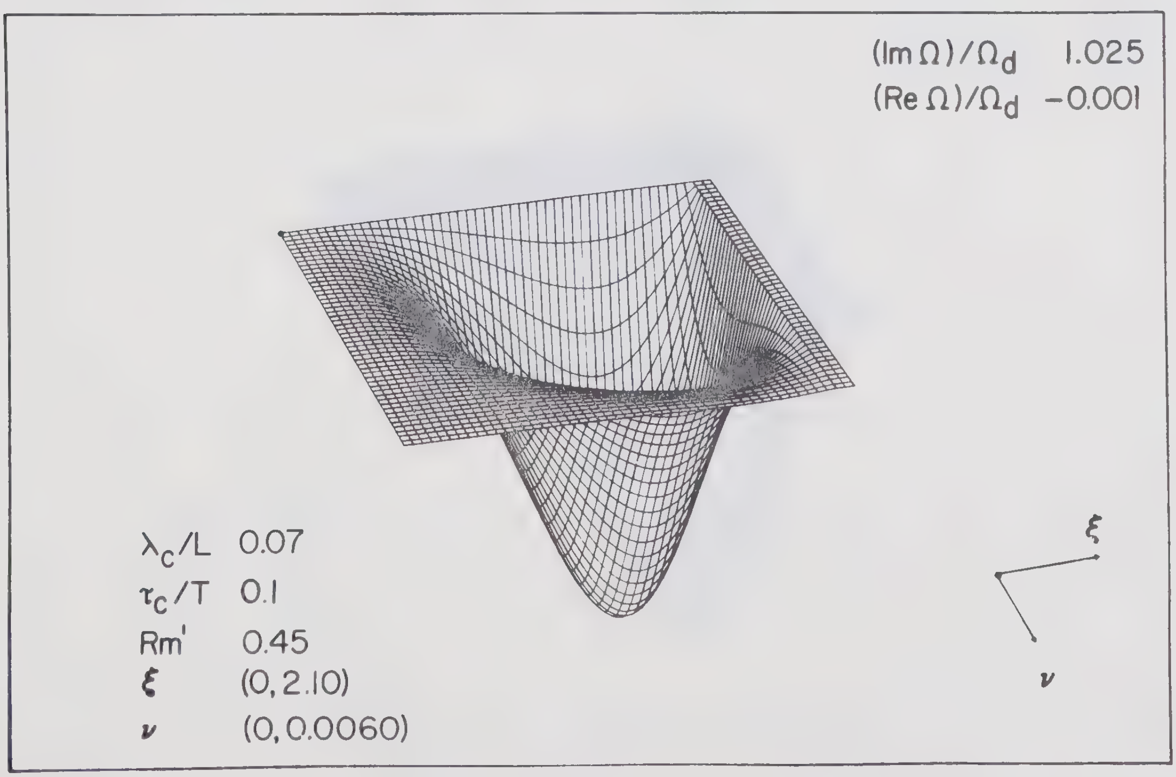


Figure 14a. Integrand of (3.165a)

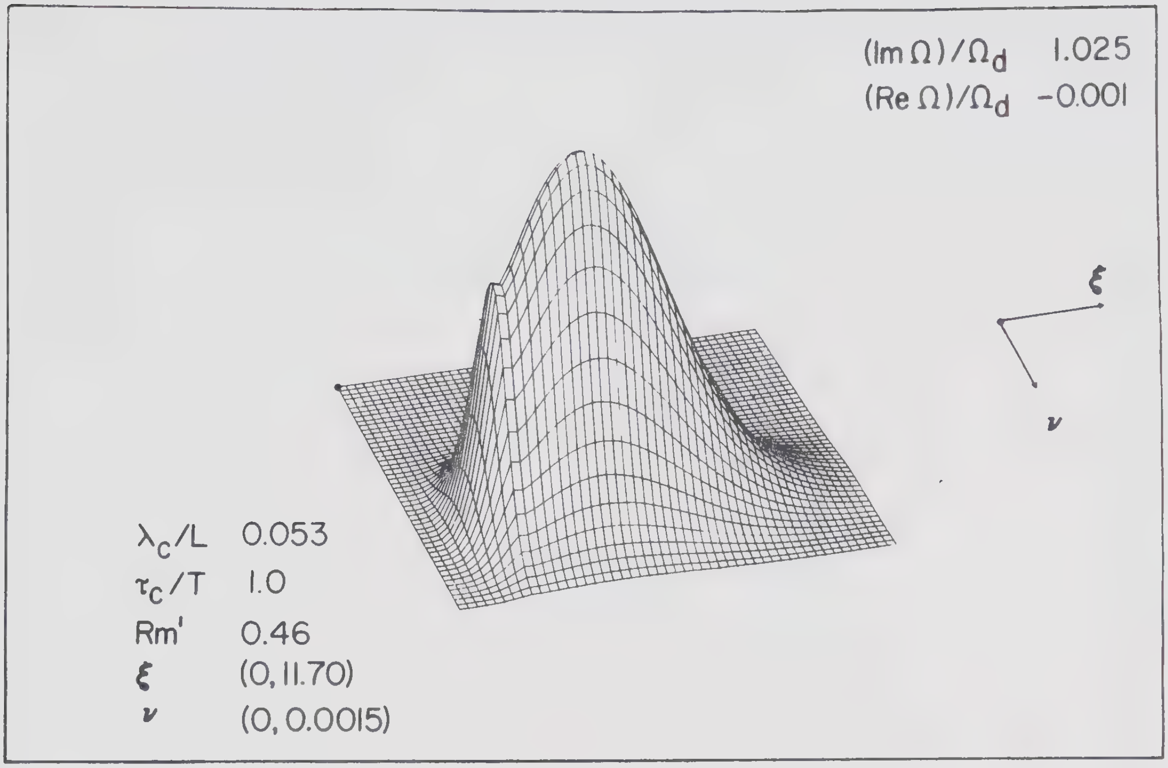


Figure 14b. Integrand of (3.165b)

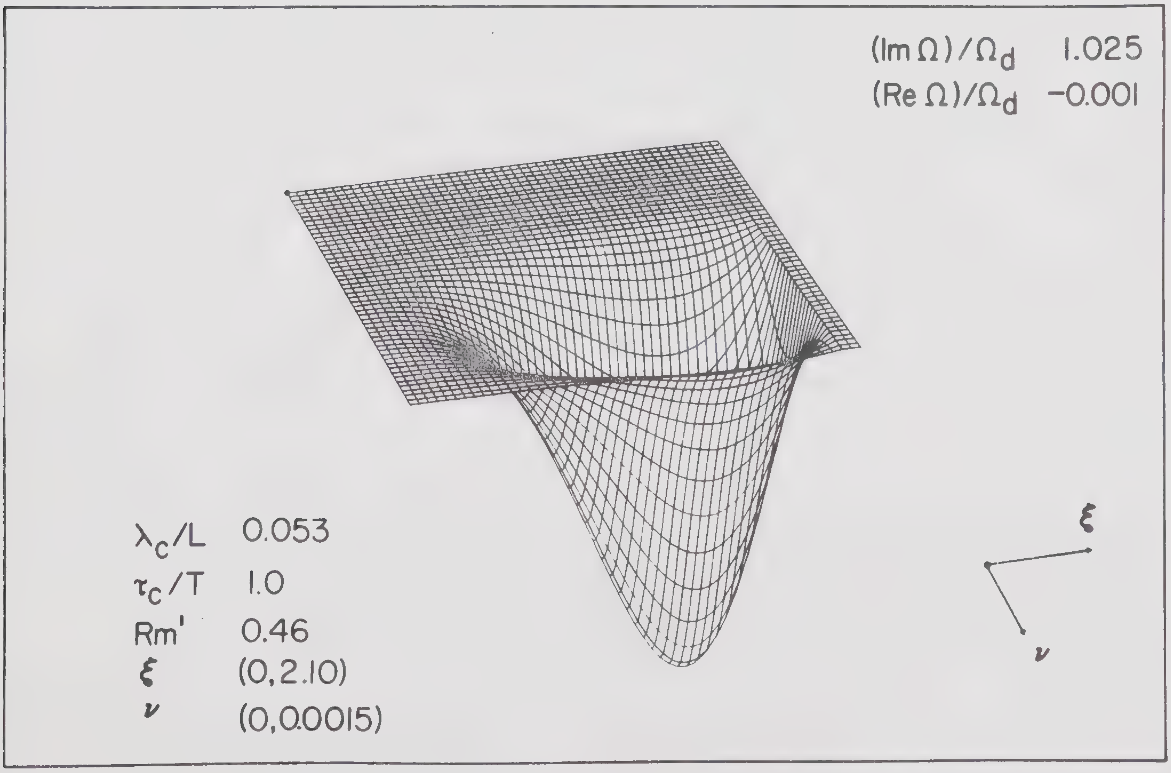
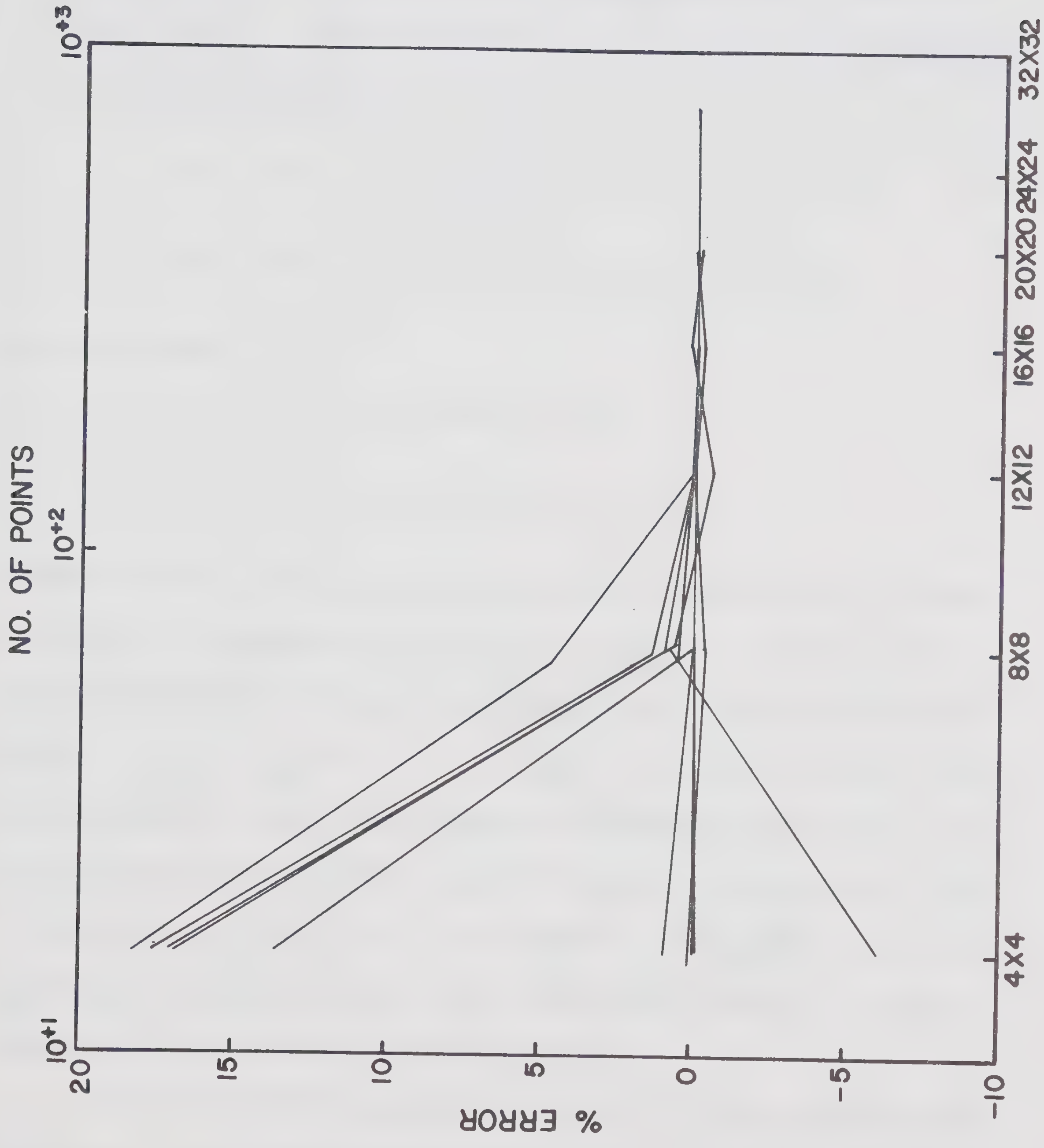


Figure 15. Typical convergence of the n-point Gaussian scheme used to evaluate the integrals in (3.165).

The plot shows percentage difference from the limiting value as a function of the number of integration points used. The upper scale shows *number of points*; the lower scale shows *grid size*.


NO. OF POINTS
10^{+1}
10^{+2}
10^{+3}
% ERROR
20
15
10
5
0
-5
-10
4X4
8X8
12X12
16X16
20X20
24X24
32X32

3.10.3 Behaviour of solutions as functions of λ_c/L, τ_c/T, and R_m'

Typical results of the calculations are shown in *Figures 16-19*. The solutions to (3.165) may be thought of as a pair of functions

$$\mathrm{Im}\ \Omega/\eta K^2 = \mathrm{Im}\ \Omega/\eta K^2\ [\lambda_c/L,\ \tau_c/T,\ R_m']$$

$$\mathrm{Re}\ \Omega/\eta K^2 \quad \mathrm{Re}\ \Omega/\eta K^2\ [\lambda_c/L,\ \tau_c/T,\ R_m']$$

The equations

$$\mathrm{Im}\ \Omega/\eta K^2 = \text{constant} = C_1$$

$$\mathrm{Re}\ \Omega/\eta K^2 = \text{constant} = C_2$$

then define a curve representing the intersection of two surfaces in the space $[\lambda_c/L,\ \tau_c/T,\ R_m']$. *Figures 16-18* show projections of this curve onto the three coordinate planes $[\lambda_c/L,\ \tau_c/T]$, $[\tau_c/T,\ R_m']$, and $[\lambda_c/L,\ R_m']$ for a fixed value of C_2 and several different values of C_1 corresponding to roughly equal logarithmic spacing of the values of $(\mathrm{Im}\ \Omega/\eta K^2)-1$. On most of the plots, λ_c/L and τ_c/T are restricted to values less than unity, in accordance with the conditions (3.147a,b).

The projection onto the $[\lambda_c/L,\ \tau_c/T]$ plane, shown in *Figure 16*, indicates that the value of λ_c/L corresponding to an oscillatory solution of (3.165) is more or less independent of τ_c/T when τ_c/T is small (i.e. when

$[(\lambda_c K)^2/q]\cdot \mathrm{Re}(\Omega/\eta K^2)$ is large, $\gtrsim 1.9$). As $\mathrm{Im}(\Omega/\eta K^2)$ increases, solutions are found at progressively lower values of λ_c/L . On the other hand, *Figure 17*, which shows the projection onto the $[\tau_c/T, R_m']$ plane, indicates that as $\mathrm{Im}(\Omega/\eta K^2)$ increases, *so must* R_m' . In addition, for each value of $\mathrm{Re}(\Omega/\eta K^2)$, R_m' reaches a *minimum* at $\tau_c/T \gtrsim 0.3$ when $(\mathrm{Im}\,\Omega/\eta K^2)-1 \simeq \mathrm{Re}(\Omega/\eta K^2)$. The value of R_m' at this minimum *decreases* with $\mathrm{Re}(\Omega/\eta K^2)$, as may be seen from *Figure 19*.

Figure 17 also indicates that at a given value of $\mathrm{Re}(\Omega/\eta K^2)$, no solutions to (3.165) exist when

$$\begin{aligned} R_m' &< (R_m')_c[\tau_c/T;\ \mathrm{Re}(\Omega/\eta K^2)] \\ \text{or}\quad \tau_c/T &< (\tau_c/T)_c[R_m';\ \mathrm{Re}(\Omega/\eta K^2)] \end{aligned} \tag{3.168}$$

where the functions $(R_m')_c$ and $(\tau_c/T)_c$ are defined by the *lower envelope* of the plotted curves when $\tau_c/T < 0.3$, and by the *minimum in* R_m' when $\tau_c/T \gtrsim 0.3$. As might be expected from the way in which λ_c/L and τ_c/T are interrelated, (3.168) is equivalent to the statement that there exist no solutions to (3.165) at a given value of $\mathrm{Re}(\Omega/\eta K^2)$ when

$$\begin{aligned} R_m' &< (R_m')_c[\lambda_c/L;\ \mathrm{Re}(\Omega/\eta K^2)] \\ \text{or}\quad \lambda_c/L &< (\lambda_c/L)_c[R_m';\ \mathrm{Re}(\Omega/\eta K^2)] \end{aligned} \tag{3.169}$$

where $(R_m')_c$ and $(\lambda_c/L)_c$ are defined by the dotted

cut-off curve in *Figure 18*.

Figure 18, which shows the projection of the solution curves onto the $[\lambda_c/L, R'_m]$ plane, gives a clear illustration of the existence of a *minimum in* R'_m for a given value of $\mathrm{Re}(\Omega/\eta K^2)$. For the value chosen in the plot $[\mathrm{Re}(\Omega/\eta K^2) = 0.001]$ the minimum in R'_m occurs at $\lambda_c/L \simeq 0.25$. The same result is obtained at different values of $\mathrm{Re}(\Omega/\eta K^2)$, over a fairly wide range, as may be seen from *Figure 19*.

As $\mathrm{Re}(\Omega/\eta K^2)$ is varied, the cut-off curve in *Figure 18* retains its general shape, and the minimum stays at the same value of λ_c/L. In addition, the curves of constant $\mathrm{Im}(\Omega/\eta K^2)$ do not alter greatly. The cut-off merely moves down in R'_m with decreasing $\mathrm{Re}(\Omega/\eta K^2)$, allowing the curves of constant $\mathrm{Im}(\Omega/\eta K^2)$ to extend further toward the left of the diagram.

Figure 19 is a plot of curves of constant R'_m and curves of constant λ_c/L as functions of $(\mathrm{Im}\ \Omega/\eta K^2)-1$ and $\mathrm{Re}(\Omega/\eta K^2)$ at a fixed value of $(\lambda_c K)^2/q$. The only curve of constant λ_c/L shown is the one which passes through the points corresponding to the minimum R'_m for each value of $\mathrm{Re}(\Omega/\eta K^2)$. Other curves of constant λ_c/L run roughly parallel to the one shown, with λ_c/L increasing toward lower values of $(\mathrm{Im}\ \Omega/\eta K^2)-1$.

Figure 20 is a plot of curves of constant $\mathrm{Re}(\Omega/\eta K^2)$ and curves of constant λ_c/L as functions of $(\mathrm{Im}\ \Omega/\eta K^2)-1$

and R'_m at the same fixed value of $(\lambda_c K)^2/q$ as in *Figure 19* $[(\lambda_c K)^2/q = 30]$. Only two curves of constant λ_c/L are shown - those for $\lambda_c/L = 0$ and $\lambda_c/L = 1$. Because of condition (3.147a), all acceptable solutions will lie between these two curves.

It is interesting to compare *Figure 20* with *Figure 4*. Clearly, when $\lambda_c/L = 0$ and $\mathrm{Re}(\Omega/\eta K^2) = 0$, the relationship between R'_m and $(\mathrm{Im}\ \Omega/\eta K^2)-1$ is the same on both plots - at least for R'_m greater than the minimum shown on *Figure 20*. However, at larger values of λ_c/L , *Figure 20* gives only a *single* pair of values $[R'_m;(\mathrm{Im}\ \Omega/\eta K^2)-1]$ for each value of λ_c/L , in contrast to the *infinite range* of values shown in *Figure 4*. *Figure 20* thus shows how the solutions of the eigenvalue problem for oscillatory mean fields form a discrete subset of the solutions for non-oscillatory mean fields in the limit $\mathrm{Re}\ \Omega \to 0$, as suggested in section 3.4.3.

Figure 20 clearly illustrates the existence of a minimum in R'_m as $\mathrm{Im}(\Omega/\eta K^2)$ is varied and $\mathrm{Re}(\Omega/\eta K^2)$ is held constant. It is apparent that the value of R'_m at the minimum increases monotonically with $\mathrm{Re}(\Omega/\eta K^2)$. From *Figure 17* it may also be seen that the minimum value of R'_m increases monotonically as $(\lambda_c K)^2/q$ *decreases*. The slope of the envelope curve in *Figure 17* shows that, for $\tau_c/T < 0.3$,

$$(R'_m)^2_{min} \propto (\tau_c/T)^{-1} \tag{3.170}$$

Figure 16. Solutions of (3.165) - projection onto the $[\lambda_c/L, \tau_c/T]$ plane.

Each curve shown is the projection onto the $[\lambda_c/L, \tau_c/T]$ plane of a curve in $[\lambda_c/L, \tau_c/T, R'_m]$ space defined by the intersection of the two surfaces

$$\mathrm{Im}\ \Omega/\eta K^2 = \text{constant} = C_1$$

$$\mathrm{Re}\ \Omega/\eta K^2 = \text{constant} = C_2$$

Curves are plotted for several different values of C_1 at a fixed value of C_2. In the diagram, $\Omega_d \equiv \eta K^2$.

The dots on the fourth curve $[C_1 = 1.025]$ indicate the values of λ_c/L and τ_c/T used in plotting the surfaces shown in *Figures 10-14*.

Projections onto the two remaining coordinate planes are shown in *Figures 17 and 18*.

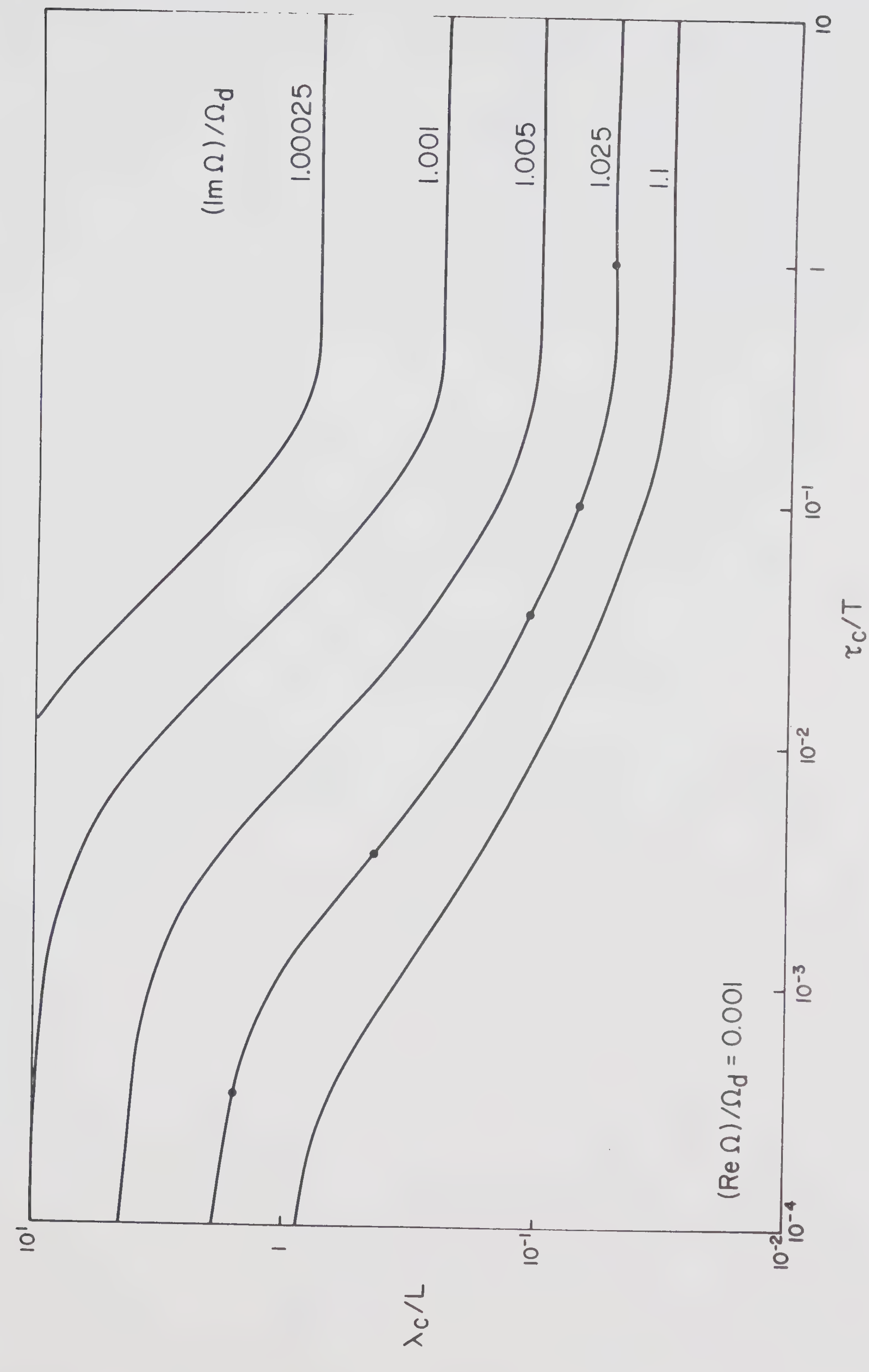

λc/L
10¹
1
10⁻¹
10⁻²
10⁻⁴
10⁻³
10⁻²
10⁻¹
1
10
τc/T
(Im Ω)/Ωd
1.00025
1.001
1.005
1.025
1.1
(Re Ω)/Ωd = 0.001

Figure 17. Solutions of (3.165) - projection onto the $[\tau_c/T, R'_m]$ plane.

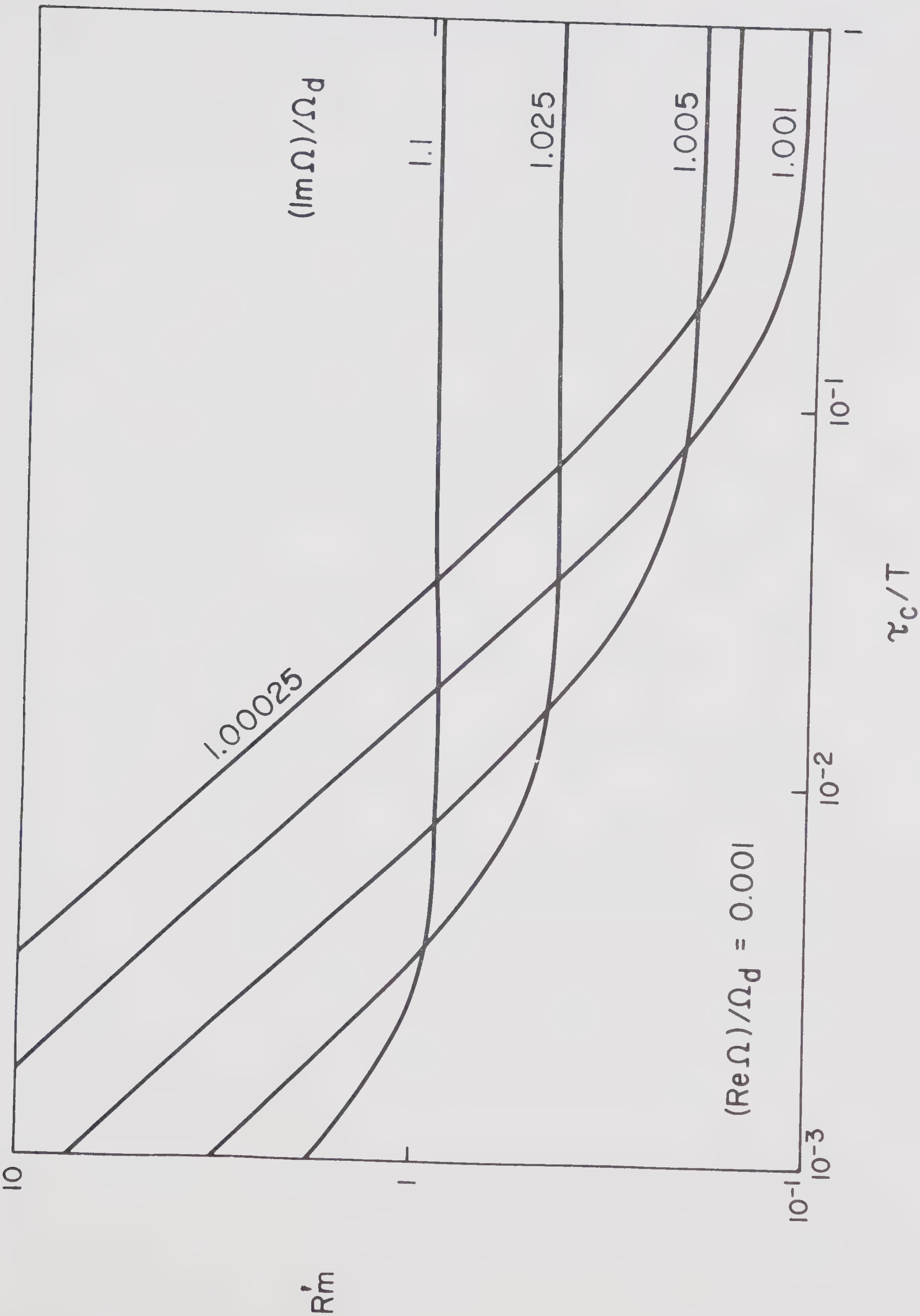
Each curve shown is the projection onto the $[\tau_c/T, R'_m]$ plane of a curve in $[\lambda_c/L, \tau_c/T, R'_m]$ space defined by the intersection of the two surfaces

$$\mathrm{Im}\ \Omega/\eta K^2 = \text{constant} = C_1$$

$$\mathrm{Re}\ \Omega/\eta K^2 = \text{constant} = C_2$$

Curves are plotted for several different values of C_1 at a fixed value of C_2. In the diagram, $\Omega_d \equiv \eta K^2$.

Projections onto the two remaining coordinate planes are shown in *Figures 16 and 18*.



$(\mathrm{Im}\,\Omega)/\Omega_d$
1.1
1.025
1.005
1.001
1.00025
$(\mathrm{Re}\,\Omega)/\Omega_d = 0.001$
10
1
10^{-1}
10^{-3}
10^{-2}
10^{-1}
1
τ_c/T
R'_m

Figure 18. Solutions of (3.165) - projection onto the $[\lambda_c/L, R'_m]$ plane.

Each curve shown is the projection onto the $[\lambda_c/L, R'_m]$ plane of a curve in $[\lambda_c/L, \tau_c/T, R'_m]$ space defined by the intersection of the two surfaces

$$\mathrm{Im}\ \Omega/\eta K^2 = \text{constant} = C_1$$

$$\mathrm{Re}\ \Omega/\eta K^2 = \text{constant} = C_2$$

Curves are plotted for several different values of C_1 at a fixed value of C_2. In the diagram, $\Omega_d \equiv \eta K^2$.

Projections onto the two remaining coordinate planes are shown in *Figures 16 and 17*.

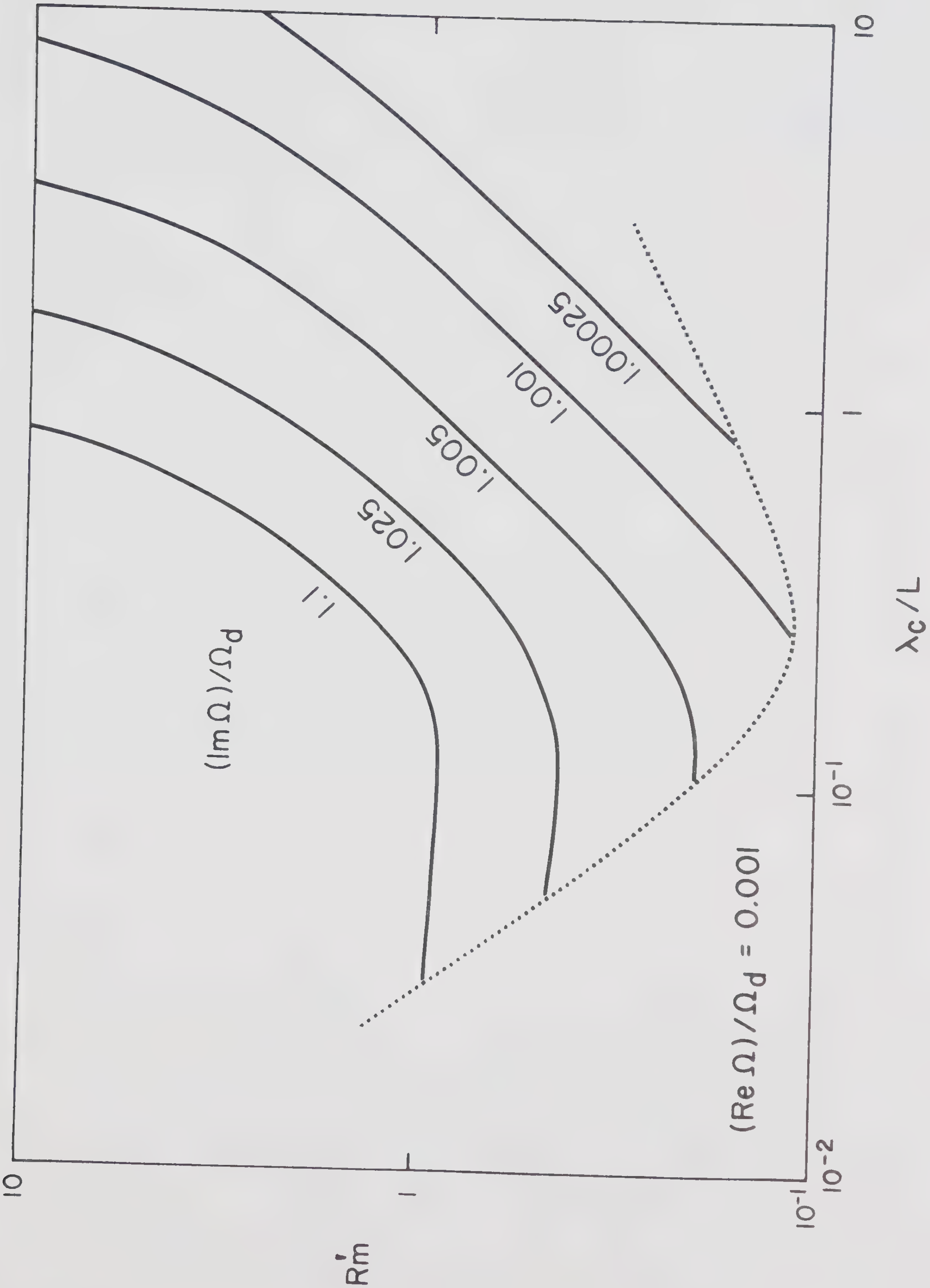

Rm
10
1
10⁻¹
10⁻²
10⁻¹
1
10
λc/L
(Im Ω)/Ωd
1.1
1.025
1.005
1.001
1.000025
(Re Ω)/Ωd = 0.001

Figure 19. Solutions of (3.165) - projection onto the [Re $\Omega/\eta K^2$, (Im $\Omega/\eta K^2$)-1] plane.

Each of the solid curves is the projection onto the [Re $\Omega/\eta K^2$, (Im $\Omega/\eta K^2$)-1] plane of the intersection of two surfaces

$$R'_m = \text{constant} = C'_1$$

$$\tau_c \eta K^2 = \text{constant} = C'_2$$

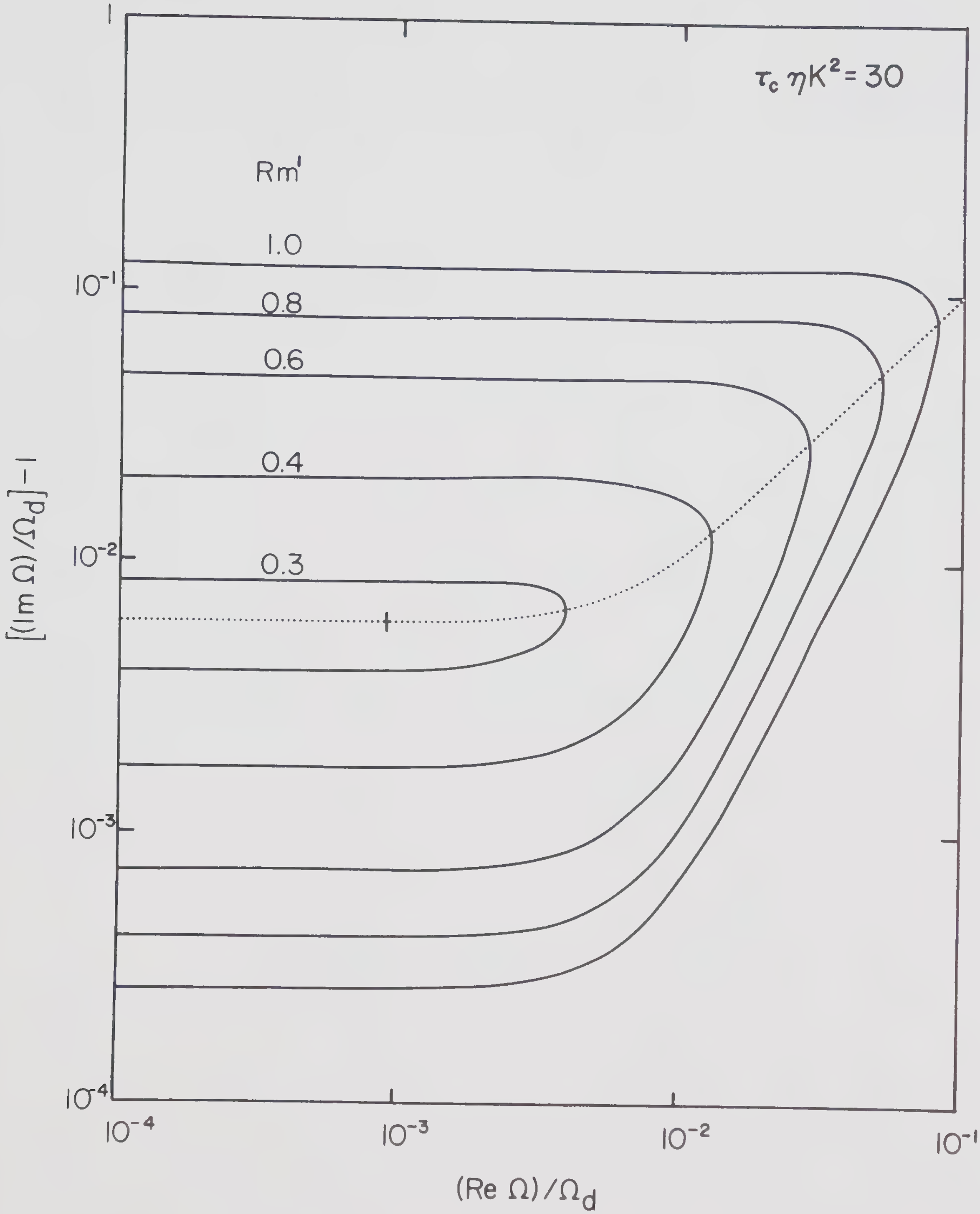
Curves are plotted for several different values of C'_1 at a fixed value of C'_2. In the diagram, $\Omega_d \equiv \eta K^2$.

The dotted curve is the projection onto the same plane of the intersection of the two surfaces

$$\lambda_c/L = \text{constant} = C'_3$$

$$\tau_c \eta K^2 = \text{constant} = C'_2$$

C'_3 is chosen in such a way that the curve passes through the points corresponding to the *minimum value of* R'_m at each value of Re $\Omega/\eta K^2$. Curves for other values of C'_3 run roughly parallel to the dotted curve, with C'_3 increasing toward lower values of (Im $\Omega/\eta K^2$)-1.



$\tau_c \eta K^2 = 30$
Rm'
1.0
0.8
0.6
0.4
0.3
$[(\mathrm{Im}\,\Omega)/\Omega_d]-1$
$(\mathrm{Re}\,\Omega)/\Omega_d$
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}

Figure 20. Solutions of (3.165) - projection onto the $[R'_m, (\mathrm{Im}\ \Omega/\eta K^2)-1]$ plane.

Each of the solid curves is the projection onto the $[R'_m, (\mathrm{Im}\ \Omega/\eta K^2)-1]$ plane of the intersection of two surfaces

$$\mathrm{Re}\ \Omega/\eta K^2 = \text{constant} = C_2$$

$$\tau_c \eta K^2 = \text{constant} = C'_2$$

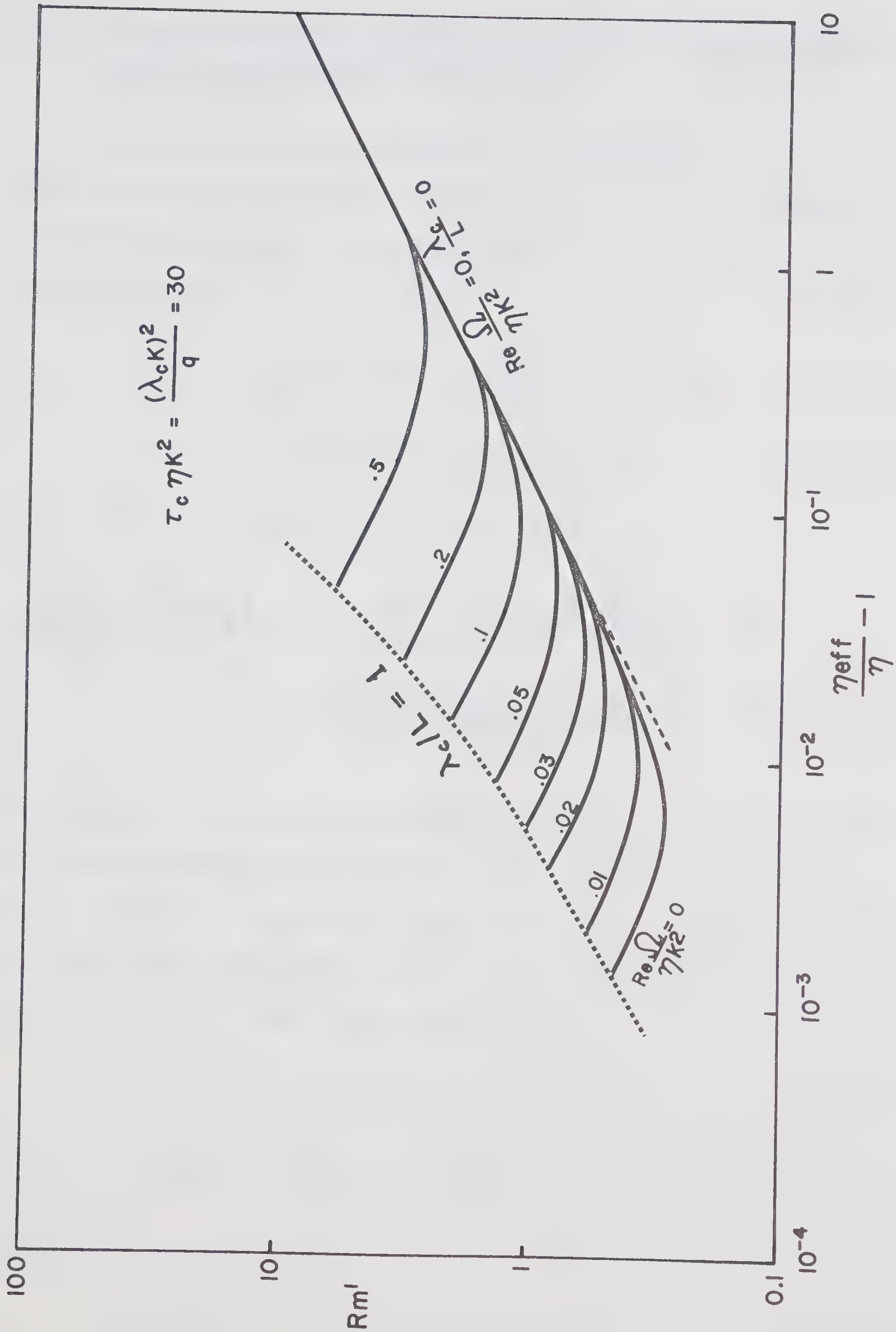
Curves are plotted for several different values of C_2 at the value of C'_2 used in *Figure 19* [i.e. $C'_2 = 30$]. In the diagram, (η/η_{eff}) has been written in place of $\mathrm{Im}\ \Omega/\eta K^2$ *(see section 3.6.2, equation 3.86)*.

The dotted curves are the projections onto the same plane of the intersection of the surfaces

$$\lambda_c/L = \text{constant} = C'_3$$

$$\tau_c \eta K^2 = \text{constant} = C'_2$$

for $C'_2 = 30$ and $C'_3 = 0, 1$.



$\tau_c \eta K^2 = \frac{(\lambda_c K)^2}{q} = 30$
.5
.2
.1
.05
.03
.02
.01
$Re \frac{\Omega}{\eta K^2} = 0, \frac{\lambda_c}{L} = 0$
$Re \frac{\Omega}{\eta K^2} = 0$
$\lambda_c/L = 1$
Rm'
100
10
1
0.1
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10
$\frac{\eta_{eff}}{\eta} - 1$

3.10.4 The existence of acceptable slowly-decaying, long-period, oscillatory mean fields

Although *initial condition II* does not lead to a restriction like (3.148) on $(\lambda_c K)^2/q$, it is of interest to see whether such a condition can be satisfied. From *Figures 16 and 17* we see that

$$R'_m = \beta_1 [\mathrm{Im}(\Omega/\eta K^2), \mathrm{Re}(\Omega/\eta K^2)]/\tau_c \eta K^2 \qquad (3.171)$$

$$\lambda_c K \rightarrow \beta_2 [\mathrm{Im}(\Omega/\eta K^2), \mathrm{Re}(\Omega/\eta K^2)] \qquad (3.172)$$

when $(\lambda_c K)^2/q = \eta\tau_c K^2 << 1$. Therefore

$$\lim_{(\lambda_c K)^2/q \to 0} \left\{ R'_m/q \right\} = \lim_{(\lambda_c K)^2/q \to 0} \left\{ R'_m \frac{(\tau_c \eta K^2)}{(\lambda_c K)^2} \right\}$$

$$= \frac{\beta_1}{\beta_2^2} [\mathrm{Im}\, \Omega/\eta K^2, \mathrm{Re}\, \Omega/\eta K^2] \qquad (3.173)$$

This limit will give the smallest possible value of R'_m/q for given values of $\mathrm{Im}(\Omega/\eta K^2)$ and $\mathrm{Re}(\Omega/\eta K^2)$.

Figure 21 shows how R'_m/q and λ_c/L depend on $\mathrm{Im}(\Omega/\eta K^2)$ and $\mathrm{Re}(\Omega/\eta K^2)$ when $\eta\tau_c K^2$ is small. For $\eta\tau_c K^2 \lesssim 5 \times 10^{-4}$, and $\mathrm{Re}(\Omega/\eta K^2) \lesssim 100$,

$$\lambda_c/L \simeq 0.35\{(\mathrm{Im}\ \Omega/\eta K^2)-1\}^{-1/2} \qquad (3.174a)$$

$$R_m/q \simeq 0.78\{(\mathrm{Im}\ \Omega/\eta K^2)-1\}^{-1/2} \qquad (3.174b)$$

so that

$$(\lambda_c K)(R'_m/q) = 2\pi(\lambda_c/L)(R'_m/q) \simeq 1.7 \qquad (3.175)$$

In general, there is very little dependence on $\mathrm{Re}(\Omega/\eta K^2)$.

It may be seen from *Figure 21* that the condition (3.148) *can* in fact be satisfied by solutions to (3.165) which also satisfy (3.147) and (3.150). In other words, there exist solutions to (3.165) for which

a) the wavelength is long compared with the correlation length of the turbulence $[\lambda_c/L < 1]$

b) the decay time is long compared with the correlation time of the turbulence $[\tau_c/T_d \ll 1$ - i.e. $\eta\tau_c K^2 \ll 1]$

c) the first order smoothing approximation is valid $[R_m' < q$, for q large$]$

d) the period is long compared with the correlation time of the turbulence $[\tau_c/T \ll 1$ - this condition holds since the solutions depend very little on $\mathrm{Re}(\Omega/\eta K^2)]$

The range in which solutions of this type can occur is very limited, as may be seen from *Figure 21*. Clearly, for the conditions (a)-(d) all to be satisfied, we must have $1.13 \lesssim \mathrm{Im}(\Omega/\eta K^2) \lesssim 2.6$.

It is never possible to satisfy (3.148) and (3.150) simultaneously when q is small. In this limit, (3.150) requires that $R_m' \ll 1$, while (3.148) requires that $q/(\lambda_c K)^2 \gg 1$. But, for a given value of $\mathrm{Re}(\Omega/\eta K^2)$, (3.170) implies that

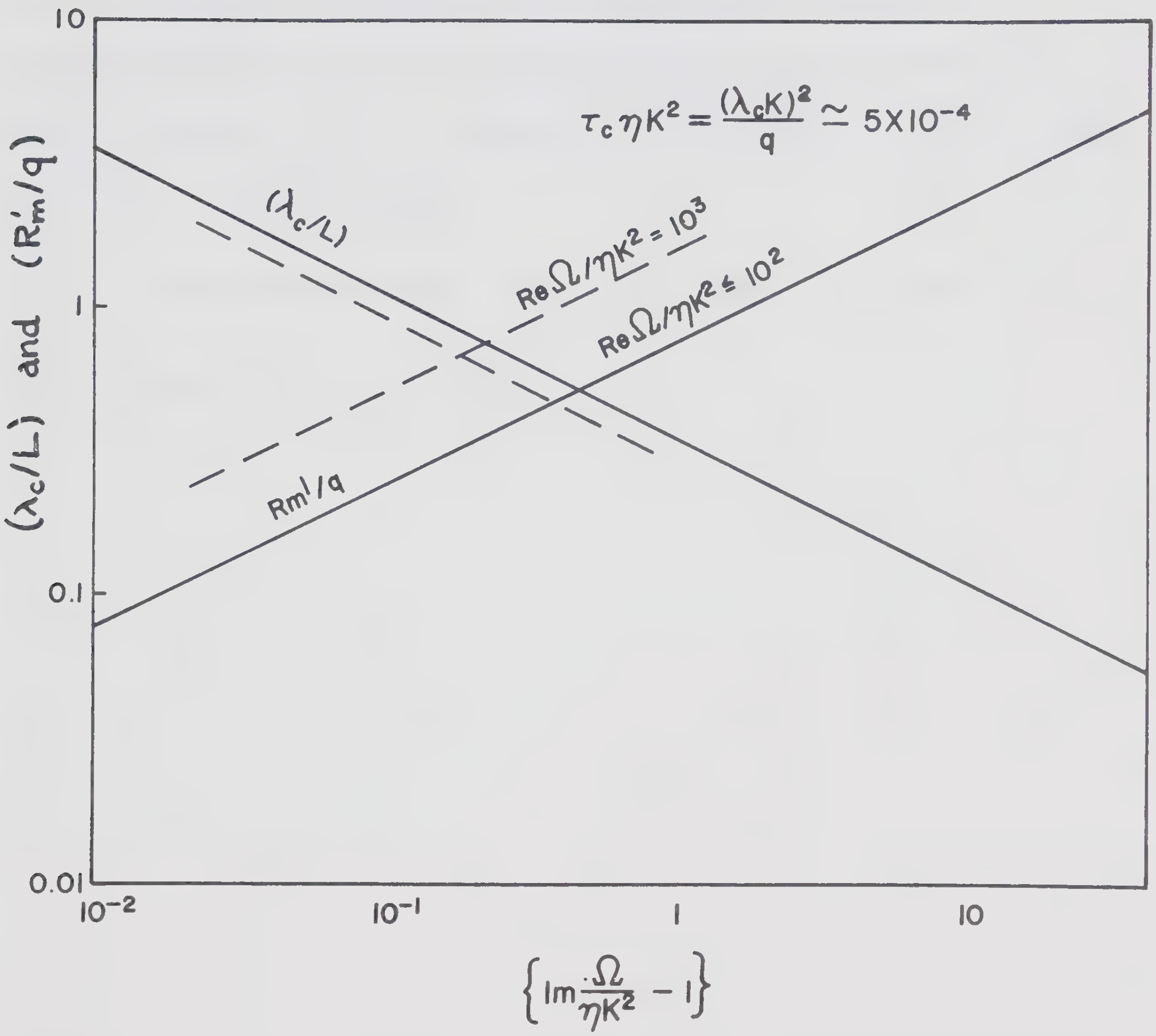


Figure 21. Solutions of (3.165) - (λ_c/L) and (R_m'/q) as functions of $(\mathrm{Im}\,\Omega/\eta K^2)-1$ in the limit of small $\tau_c \eta K^2$.

The solid curves show the behaviour of (λ_c/L) and (R_m'/q) when $\mathrm{Re}\,\Omega/\eta K^2 < 10^2$. At larger values of $\mathrm{Re}\,\Omega/\eta K^2$, (λ_c/L) and (R_m'/q) depend on *both* $\mathrm{Re}\,\Omega/\eta K^2$ and $\mathrm{Im}\,\Omega/\eta K^2$, as indicated by the dashed curves [$\mathrm{Re}\,\Omega/\eta K^2 = 10^3$].

$$(R'_m)^2_{min} \propto \frac{(T/\tau_c)}{Re\,\Omega/\eta K^2} \propto \frac{q}{(\lambda_c K)^2} \tag{3.176}$$

It may be seen from *Figure 17* that the constant of proportionality on the right hand side of (3.176) is approximately 12.4 . Hence, in the limit as $q \to 0$ but $q/(\lambda_c K)^2$ remains large,

$$[q \to 0;\ q/(\lambda_c K)^2 \text{ large}] \qquad R'_m \gg \sqrt{12.4} \simeq 3.5 \tag{3.177}$$

in contradiction to (3.150).

3.11 The kinematic dynamo problem

3.11.1 The two-dimensional integral technique and turbulence without PT-invariance

For the case of decaying mean fields, it has been found that the two-dimensional integration scheme used in treating *initial condition II* is generally easier to handle than the one-dimensional scheme used in dealing with *initial condition I*. Although the principal difficulty encountered with the one-dimensional integration will not arise in the case of *growing* mean fields, it may well be that the two-dimensional technique is better suited to a general study of dynamo action in turbulent fluids.

The irregular behaviour exhibited by the function $\mathrm{Re}\,\Theta$ when the mean field is rapidly decaying $[\mathrm{Im}(\Omega/\eta K^2) > 1]$ is not present when the mean field is *growing or slowly decaying* $[\mathrm{Im}(\Omega/\eta K^2) < 1]$, as may be seen from *Figure 1*. It remains to be shown that the evaluation of the integral $I_{ij}^{(2)}$ defined in (2.89) can be carried out easily by the two-dimensional integration technique when helicity is present.

The simplest possible case of turbulence which is not PT-invariant is described by the spectrum tensor *(Batchelor, 1953, p. 43; Moffatt, 1970a)*

$$\Phi_{ij} = \frac{E(k,\omega)}{4\pi k^4}\left\{k^2\delta_{ij} - k_i k_j\right\} + i\epsilon_{ijl}\frac{F(k,\omega)}{8\pi k^4}k_l \qquad (3.178)$$

From (3.178) and (2.89'),

$$I_{ij}^{(2)} = i\epsilon_{i\ell m} K_\ell \iint\limits_{\underset{\sim}{k}\,\omega} \frac{F(k,\omega)}{4\pi k^4} \frac{k_j k_m}{i(\omega+\Omega)+\eta(\underset{\sim}{k}+\underset{\sim}{K})^2}\, d\underset{\sim}{k}\, d\omega \qquad (3.179)$$

Taking $\underset{\sim}{K}$ to define the z-axis in $\underset{\sim}{k}$-space, as was done in (3.18),

$$I_{ij}^{(2)} = \frac{i}{8}\epsilon_{i\ell j} K_\ell \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk\, \frac{F(k,\omega)}{k^4}\cdot 2k^4 \cdot$$

$$\cdot \int_0^{\pi} \frac{\sin^3\theta\, d\theta}{i(\omega+\Omega)+\eta(k^2+K^2+2kK\cos\theta)}$$

$$= \frac{i}{8}\epsilon_{i\ell j} K_\ell K^5 \int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\xi\, \frac{F(K\xi, \eta K^2\nu + \mathcal{R}e\,\Omega)}{(K\xi)^4}\, \Theta\!\left(\xi,\nu; \mathcal{I}m\frac{\Omega}{\eta K^2}\right) \qquad (3.180)$$

It follows, therefore, that the integrals to be evaluated are of precisely the same form as those already treated in the case of PT-invariant turbulence.

3.11.2 The mean field dispersion relation

If we define

$$I^{(2)} \equiv \frac{1}{8} K^5 \int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\xi\, \frac{F(K\xi, \eta K^2\nu + \mathcal{R}e\,\Omega)}{(K\xi)^4}\, \Theta\!\left(\xi,\nu; \mathcal{I}m\frac{\Omega}{\eta K^2}\right) \qquad (3.181)$$

then, from (3.180),

$$I_{ij}^{(2)} = -i\epsilon_{ij\ell} K_\ell\, I^{(2)}(\underset{\sim}{K},\Omega) \qquad (3.182)$$

and

$$\{\mathcal{R}e\,\underset{\approx}{I}^{(2)}\,\mathcal{I}m\,\underset{\approx}{I}^{(2)}\}_{ij} = \{\mathcal{I}m\,\underset{\approx}{I}^{(2)}\,\mathcal{R}e\,\underset{\approx}{I}^{(2)}\}_{ij}$$

$$= -\epsilon_{ilm}\epsilon_{mnj}K_l K_n\,\mathcal{R}e\,I^{(2)}\,\mathcal{I}m\,I^{(2)}$$

$$= \{K^2\delta_{ij} - K_i K_j\}\,\mathcal{R}e\,I^{(2)}\,\mathcal{I}m\,I^{(2)} \tag{3.183}$$

Substituting (3.183) into (2.91), we see that the dispersion relation for the mean field is

$$\det(\underset{\approx}{a}\underset{\approx}{a} + \underset{\approx}{b}\underset{\approx}{b}) = 0 \tag{3.184}$$

where

$$a_{ij} = -\mathcal{I}m\{\epsilon_{ijl}K_l I^{(2)} + \delta_{ij}[\Omega - i\eta K^2 - iI^{(1)}]\} \tag{3.185a}$$

$$b_{ij} = \mathcal{R}e\{\epsilon_{ijl}K_l I^{(2)} + \delta_{ij}[\Omega - i\eta K^2 - iI^{(1)}]\} \tag{3.185b}$$

When (3.185) is substituted into (3.184) and the determinant is expanded, the dispersion relation is found to have three roots, specified by

$$i\Omega + \eta K^2 + I^{(1)} = 0 \tag{3.186a}$$

$$i\Omega + \eta K^2 + I^{(1)} = \mp K I^{(2)} \tag{3.186b,c}$$

As pointed out by *Moffatt (1970a)*, these roots correspond to *normal decay (3.186a)*, *enhanced decay (3.186b)*, and *retarded decay (or growth) (3.186c)*. (3.186c), the equation which leads to growing wave solutions, and hence to dynamo action, corresponds to the lower sign in (3.186b,c).

3.11.3 The possibility of a "sporadic helicity" dynamo

Consideration of the growing wave solutions of (3.186c) leads to a situation in which decaying wave solutions corresponding to *initial condition II* may be important. A nonstationary kinematic dynamo of the *sporadic* type discussed in section 1.4.6 can be constructed by allowing the helicity of the turbulence to vary with time. The helicity is initially "turned on" for a period long enough to allow the effects of initial conditions to die away. The growing wave solutions to (3.186c) can then be followed as $F(k,\omega)$, and hence $I^{(2)}$, is allowed to vary slowly with time. When $I^{(2)}$ goes to zero, equation (3.186c) reduces to equation (3.39), corresponding to *initial condition II*. The mean field will then decay in the manner described above in section 3.10, until such time as the helicity is "turned on" again.

3.12 Summary of Chapter 3

This chapter is concerned with the effects of PT-invariant turbulence on large scale magnetic fields. Much of the work described was carried out jointly by the present author and *Dr. K.D. Aldridge (Gilliland and Aldridge, 1973)*.

In *section 3.2* it is proved that stationary, homogeneous turbulence whose average properties are invariant under space-time inversion *(PT-invariant turbulence)* cannot support dynamo action in an incompressible fluid, in the *first order smoothing approximation* (see *section 2.5.4* for a definition of first order smoothing). This result is a generalization of a theorem due to *Krause and Roberts (1973)*. It directly contradicts the work of *Lerche and Low (1971)*.

Sections 3.3-3.10 present a detailed study of the decay of *wave* mean fields in the presence of PT-invariant turbulence. Two initial conditions are studied:

I: the initial turbulent component of the magnetic field is not correlated with the turbulent velocity - i.e. $\overline{u_j'(\underset{\sim}{x},t)B_k'(\underset{\sim}{x}',t_o)} = 0$ for all choices of $(\underset{\sim}{x},\underset{\sim}{x}',t)$. *(A less restrictive condition which leads to the same result is described in section 3.3.2.)*

II: the initial turbulent component of the magnetic field is correlated with the turbulent velocity in

such a way that $\mathrm{curl}[\,\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}\,](\underset{\sim}{x},t_o) \propto \overline{\underset{\sim}{B}}(\underset{\sim}{x},t_o)$ for all choices of $\underset{\sim}{x}$. *(See section 3.3.3.)*

In the first order smoothing approximation, spatially periodic mean magnetic fields can exist in an infinite turbulent medium only if the two-point, two-time correlation tensor of the turbulence falls off at least exponentially with time displacement τ . If the correlation tensor has an exponential dependence on τ , spatially periodic mean fields can exist only if the correlation time of the turbulence, τ_c , is shorter than the *effective decay time* of the mean field, $T_d^{eff} = [\eta_{eff}K^2]^{-1}$. η_{eff} is the *turbulent magnetic diffusivity*, and K the wave number of the mean magnetic field. *(See section 3.7.3.)*

In *sections 3.6-3.10, isotropic Gaussian turbulence* is studied as an example of turbulence in which the correlation tensor falls off more rapidly with τ than $\exp[-|\tau|/\tau_c]$. Several restrictions on the parameters of the mean field and the turbulence arise.

a) When *initial condition I* applies, spatially periodic mean fields can exist only if τ_c is shorter than βT_d^{eff} , where $\beta = \beta(\tau_c/T_d;\ \lambda_c/L)$. T_d is the natural decay time of the mean field in the absence of turbulence, λ_c the correlation length of the turbulence, and L the wavelength of the mean field.

Furthermore, spatially periodic mean fields can

exist only if the *magnetic Reynolds number of the turbulence*, R'_m, is less than a *critical value*, $(R'_m)_{crit}[\tau_c/T_d;\ \lambda_c/L]$. It can be shown that both β and $(R'_m)_{crit}$ depend only weakly on λ_c/L. [*See section 3.8.1 for the work described here.*]

b) When *initial condition II* applies, spatially periodic mean fields can exist only if $\tau_c < \beta' T_d^{eff}$, where $\beta' = \beta'[\lambda_c/L]$. There is no restriction on R'_m. [*See section 3.6.5.*]

c) The *effective magnetic diffusivity*, η_{eff}, normally depends on the parameters of the turbulence [τ_c, λ_c, R'_m], the relationship between the turbulence and the mean field [τ_c/T_d, λ_c/L], and the initial conditions [*I*, *II*]. However, both the dependence on the properties of the mean field and the dependence on initial conditions are weak, and disappear entirely in certain circumstances.

d) η_{eff} is independent of initial conditions if [*section 3.8.3*]

i. $\tau_c \ll T_d^{eff}$

<u>*and*</u> ii. the diffusion time on the length scale of the turbulence, $(\lambda_c^2/\eta) \ll$ the effective diffusion time on the length scale of the mean field, (L^2/η_{eff}).

e) When *initial condition I* applies, η_{eff} is

independent of the properties of the mean field if [*section 3.8.2*]

$$\tau_c < 0.025\ T_d^{eff}\ , \quad [\lambda_c^2/\eta < 0.01\ \tau_c]$$

$$\tau_c < 0.025\ [T_d T_d^{eff}]/[T_d - T_d^{eff}]\ , \quad [\lambda_c^2/\eta > 0.01\ \tau_c]$$

f) When *initial condition I* applies, η_{eff} changes with time, *stabilizing* at a limiting value $[\eta_{eff}]^*$ after a time T_1. $[\eta_{eff}]^*$ can be used to describe the decay of the mean field provided that $T_1 << T_d^{eff}$ [*see section 3.8.4 for a more precise statement of this condition*].

The condition on T_1 can be translated into a condition on the parameters of the turbulence and the mean field [*see section 3.8.5 and Figure 8*]. In general, T_1 must be less than two correlation times [*much less, if* $\lambda_c^2/\eta << \tau_c$] for $[\eta_{eff}]^*$ to be a meaningful parameter.

g) When *initial condition II* applies, η_{eff} is independent of the properties of the mean field if $\lambda_c << 0.16\ L$ [*see section 3.6.3*].

Restrictions (d)-(g) can be interpreted as restrictions on the usefulness of the *Rädler expansion* (3.96) as a representation of $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$. The Rädler expansion is

obtained using *initial condition I*, and only the first term of the expansion is independent of the properties of the mean field.

It should be noted that the *Rädler expansion* is not useful when the mean field oscillates with time [*section 3.9.1*]. For this reason, the validity of *Krause and Rädler's* expression for the "turbulent conductivity" appropriate to an oscillatory, decaying mean field *(Krause and Rädler, 1971, pp. 70-71)* is open to question.

Sections 3.9 and 3.10 present a study of the behaviour of decaying mean fields, periodic in space and oscillatory in time, in the presence of *Gaussian turbulence*. It is shown [*sections 3.9.4 and 3.9.5*] that if *initial condition I* applies, no mean fields of this type with

$$\left\{ (\tau_c/T_d^{eff})^2 + (\tau_c/T)^2 \right\}^{1/2} \ll 1$$

can exist unless (τ_c/T) is identically zero. T is the oscillation period of the mean field.

On the other hand, if *initial condition II* applies, spatially periodic, oscillatory mean fields for which $\tau_c \ll T$ can exist. (*Initial condition II* does not lead to the requirement that $\tau_c \ll T_d^{eff}$.) The behaviour of these fields is described in *section 3.10.3*. If the condition $\tau_c \ll T_d$ is imposed on solutions obtained using *initial condition II*, it is found that turbulence for which

$\lambda_c^2/\eta \ll \tau_c$ cannot support spatially periodic, oscillatory mean fields of the required type [*section 3.10.4*].

In *section 3.11* the effect of relaxing the condition of *PT-invariance* on the turbulence is investigated. It is found that the mean field dispersion relation for this problem, while somewhat more complicated than the one studied earlier in the chapter, still involves only integrals of the type studied in *section 3.10*. It is suggested that the numerical techniques used in *section 3.10* may well provide the most convenient method for investigating dynamo action generated by *non-PT-invariant* turbulence. The possibility of a dynamo with *sporadic helicity* is also discussed [*section 3.11.3*].

4. THE DYNAMO PROBLEM AND INHOMOGENEOUS, NONSTATIONARY TURBULENCE

4.1 Introduction

As pointed out in the last chapter, the assumption of stationary, homogeneous turbulence is a gross over-simplification in a real, finite fluid. One of the principal difficulties lies in the fact that no boundary conditions can be applied to turbulence of this sort. In a dynamo like the *geodynamo*, where boundary conditions are important, it will clearly be necessary to consider *inhomogeneous* turbulence. Furthermore, if the time behaviour of the mean magnetic field is to be described with any degree of accuracy, the turbulence will have to be *nonstationary*, particularly in the *hydromagnetic dynamo problem*.

The principal effect of introducing nonstationary, inhomogeneous turbulence into mean field electrodynamics is to make $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ depend *explicitly* on position and time, as well as on the mean fields $\overline{\underset{\sim}{u}}$ and $\overline{\underset{\sim}{B}}$. When $\overline{\underset{\sim}{B}}$ is a growing function of time, equation (2.22) gives

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_0'}\}_i = \epsilon_{ijk}\,\epsilon_{\ell mn}\,\epsilon_{npq} \cdot \tag{4.1}$$

$$\cdot \int_V d\underset{\sim}{x}' \int_{-\infty}^{t} dt'\; G_{k\ell}(\underset{\sim}{x},t;\underset{\sim}{x}',t'\,|\,\overline{\underset{\sim}{u}})\,\frac{\partial}{\partial x_m'}\left\{R_{jp}(\underset{\sim}{x},t;\underset{\sim}{x}',t')\,\overline{B}_q(\underset{\sim}{x}',t')\right\}$$

after a time sufficiently long for the effects of initial conditions on $\underset{\sim}{B}'$ to have died out. Clearly, the effects of nonstationarity and inhomogeneity of the turbulence are represented by the dependence of the correlation tensor $R_{jp}(\underset{\sim}{x},t;\underset{\sim}{x}',t')$ on each of its arguments *separately*.

We may therefore introduce nonstationarity and inhomogeneity into the kinematic dynamo problem fairly simply by choosing suitable forms for the tensor R_{jp} [in the *first order smoothing approximation*], and for higher-order correlation tensors [when equations (2.25)-(2.30) are taken into account].

4.2 Inhomogeneous, nonstationary turbulence - a survey of existing techniques

4.2.1 Locally homogeneous, quasi-stationary random functions - the *Kolmogorov* structure tensor approach

One of the most commonly used approaches to the study of inhomogeneous, nonstationary turbulence is that introduced by *Kolmogorov (1941a,b)*. In this approach, a random field $\underset{\sim}{u}'(\underset{\sim}{x},t)$ is said to be *locally homogeneous and quasi-stationary* in a region G if the distribution functions of the difference

$$u_i'(\underset{\sim}{x}+\underset{\sim}{r},\ t+\tau) - u_i'(\underset{\sim}{x},t) \tag{4.2}$$

are invariant for all choices of $(\underset{\sim}{x},t)$ in G. In other words, the difference (4.2) can be considered as a stationary, homogeneous random function in G. *(See, for example, Tatarski, 1961, p. 19ff; Yaglom, 1962, p. 93; Panchev, 1971, p. 149.)*

The principal quantity of interest in this approach is the *structure tensor*,

$$D_{ij}(\underset{\sim}{r},\tau) \equiv \overline{\{u_i'(\underset{\sim}{x}+\underset{\sim}{r},t+\tau) - u_i'(\underset{\sim}{x},t)\}\{u_j'(\underset{\sim}{x}+\underset{\sim}{r},t+\tau) - u_j'(\underset{\sim}{x},t)\}} \tag{4.3}$$

which is related to the correlation tensor by the equation

$$\begin{aligned} D_{ij}(\underset{\sim}{r},\tau) = {} & R_{ij}(\underset{\sim}{x}+\underset{\sim}{r},t+\tau;\underset{\sim}{x}+\underset{\sim}{r},t+\tau) + R_{ij}(\underset{\sim}{x},t;\underset{\sim}{x},t) \\ & - R_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}+\underset{\sim}{r},t+\tau) - R_{ji}(\underset{\sim}{x},t;\underset{\sim}{x}+\underset{\sim}{r},t+\tau) \end{aligned} \tag{4.3'}$$

As may be seen from equation (4.3'), the structure tensor approach does not provide a direct method for evaluating the correlation tensor R_{ij} . The approach is therefore not immediately useful in mean field electrodynamics.

4.2.2 The *Silverman* approach to locally stationary and homogeneous turbulence with smoothly varying mean characteristics

Probably the simplest approximation to the correlation tensor of a nonstationary, inhomogeneous random process is that suggested by *Silverman (1957; see also Tatarski, 1961, pp. 53-55)*. In this approach, the average properties of the turbulent field are assumed to vary smoothly over large length and time scales.

In *Silverman's* terminology, a random field $\underset{\sim}{u}'(\underset{\sim}{x},t)$ is said to be *locally stationary and homogeneous* if its correlation tensor can be written as

$$R_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}',t') \equiv \overline{u_i'(\underset{\sim}{x}',t')\,u_j'(\underset{\sim}{x},t)}$$

$$= f\left\{\frac{\underset{\sim}{x}+\underset{\sim}{x}'}{2}, \frac{t+t'}{2}\right\} r_{ij}(\underset{\sim}{x}'-\underset{\sim}{x}, t'-t) \qquad (4.4)$$

It is clear from (4.4) that the Silverman approach will not permit boundary conditions on $\underset{\sim}{u}'$ to be applied exactly, because of the way in which the "amplitude" f is

defined. Because of this restriction, the Silverman approach has only limited application in mean field electrodynamics.

4.2.3 The *Rädler* approach to locally stationary and homogeneous turbulence with smoothly varying mean characteristics

The principal difficulty encountered in the *Silverman* approach can be overcome by means of a Taylor series approach due to *Krause and Rädler (1971) and Rädler (1972; see also P.H. Roberts, 1971a)*. In this approach it is assumed that the correlation tensor has the form

$$R_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}+\underset{\sim}{r},t+\tau) =$$

$$= u'^2(\underset{\sim}{x},t)\, m_{ij}^{(0)}(\underset{\sim}{r},\tau) + \tau \frac{\partial}{\partial t} u'^2(\underset{\sim}{x},t)\, m_{ij}^{(11)}(\underset{\sim}{r},\tau) +$$

$$+ \underset{\sim}{r}\cdot\underset{\sim}{\nabla}\, u'^2(\underset{\sim}{x},t)\, m_{ij}^{(12)}(\underset{\sim}{r},\tau) + r_i\{\underset{\sim}{\nabla} u'^2(\underset{\sim}{x},t)\}_j\, f^{(1)}(r,\tau)$$

$$+ r_j\{\underset{\sim}{\nabla}\, u'^2(\underset{\sim}{x},t)\}_i\, f^{(2)}(r,\tau) + \cdots \qquad (4.5)$$

where

$$u'^2(\underset{\sim}{x}+\underset{\sim}{r},t+\tau) = u'^2(\underset{\sim}{x},t) + \underset{\sim}{r}\cdot\underset{\sim}{\nabla}\, u'^2(\underset{\sim}{x},t) + \tau\frac{\partial}{\partial t} u'^2(\underset{\sim}{x},t) + \cdots \qquad (4.6)$$

General symmetry considerations are then used to determine the form of the tensors $m_{ij}^{(0)}$, $m_{ij}^{(11)}$, and $m_{ij}^{(12)}$. This method is very useful when dealing with the kinematic

dynamo problem, but it is not well suited to the study of the more general hydromagnetic dynamo problem.

Rädler (1972) has derived a general expression for the correlation tensor of a turbulent velocity field in which all deviations from homogeneity and isotropy can be described in terms of the gradient of the turbulence intensity.

4.3 Inhomogeneous, nonstationary turbulence and the dynamo problem - a successive approximation technique

4.3.1 Outline of the successive approximation technique

Another method for dealing with inhomogeneous, non-stationary turbulence can be suggested which *will* be applicable to the hydromagnetic dynamo problem. This method involves treating the large scale variations of both the mean fields and the turbulence by means of a *successive approximation technique*. As an illustration, we shall apply the technique here to the kinematic dynamo problem. Later, in *Chapter 6*, we shall use the technique in an investigation of the hydromagnetic dynamo problem.

We shall assume that the fluctuating fields $\underset{\sim}{u}'$ and $\underset{\sim}{B}'$ can be represented by Fourier-Stieltjes integrals of the type

$$u_i'(\underset{\sim}{x},t) = \iint_{\underset{\sim}{k}\omega} U_{ij}(\underset{\sim}{x},t;\underset{\sim}{k},\omega)\, dZ_j(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega\tau\}} \tag{4.7a}$$

$$B_i'(\underset{\sim}{x},t) = \iint_{\underset{\sim}{k}\omega} \beta_{ij}(\underset{\sim}{x},t;\underset{\sim}{k},\omega)\, dY_j(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega\tau\}} \tag{4.7b}$$

where the tensors $U_{ij}(\underset{\sim}{x},t;\underset{\sim}{k},\omega)$ and $\beta_{ij}(\underset{\sim}{x},t;\underset{\sim}{k},\omega)$ vary with position and time on scales large compared with the correlation length and time of the turbulence.

The representations (4.7a,b) may now be substituted into the fluctuating induction equation (2.10). In the first order smoothing approximation, (2.10) reduces to

(2.13). A successive-approximation solution to (2.13) may be sought in the form

$$d\underset{\sim}{Z}(\underset{\sim}{k},\omega) = d\underset{\sim}{Z}^{(0)}(\underset{\sim}{k},\omega) + d\underset{\sim}{Z}^{(1)}(\underset{\sim}{k},\omega) + \ldots \quad (4.8a)$$

$$d\underset{\sim}{Y}(\underset{\sim}{k},\omega) = d\underset{\sim}{Y}^{(0)}(\underset{\sim}{k},\omega) + d\underset{\sim}{Y}^{(1)}(\underset{\sim}{k},\omega) + \ldots \quad (4.8b)$$

with the zero-order approximation being obtained by *neglecting the large-scale variations of all quantities in (2.13).* $\underset{\sim}{B}'$ must of course satisfy the divergence condition (2.11). For simplicity, the flow will be assumed incompressible, so that $\underset{\sim}{u}'$ satisfies (1.21).

At each stage in the successive approximation, the large scale variations of all fields (and their derivatives) must be ignored to ensure that $d\underset{\sim}{Z}$ and $d\underset{\sim}{Y}$, defined by (4.8), remain functions of $\underset{\sim}{k}$ and ω alone. Under this assumption, we may make use of equation (2.48)

$$\overline{dZ_i^*(\underset{\sim}{k},\omega)\, dZ_j(\underset{\sim}{k},\omega)} = \Phi_{ij}(\underset{\sim}{k},\omega)\,\delta(\underset{\sim}{k}-\underset{\sim}{k}')\,\delta(\omega-\omega')\,d\underset{\sim}{k}\,d\underset{\sim}{k}'\,d\omega\,d\omega' \quad (4.9)$$

to obtain the expansion

$$\phi_{ij}(\underset{\sim}{k},\omega) = \phi_{ij}^{(00)}(\underset{\sim}{k},\omega) + \phi_{ij}^{(01)}(\underset{\sim}{k},\omega) + \phi_{ij}^{(10)}(\underset{\sim}{k},\omega) + \cdots \quad (4.10)$$

where

$$\overline{dZ_i^{(m)*}(\underset{\sim}{k},\omega)\, dZ_j^{(n)}(\underset{\sim}{k}',\omega')} \quad (4.11)$$

$$= \Phi_{ij}^{(mn)}(\underset{\sim}{k},\omega)\,\delta(\underset{\sim}{k}-\underset{\sim}{k}')\,\delta(\omega-\omega')\,d\underset{\sim}{k}\,d\underset{\sim}{k}'\,d\omega\,d\omega'$$

4.3.2 Solution of the fluctuating induction equation

Writing out the first few terms of the successive approximation solution in detail, we have, from (1.21) and (2.11),

$$i\underset{\sim}{k}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} = 0 \tag{4.12a}$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(n)} = -\underset{\sim}{\nabla}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(n-1)} \quad , \quad n\geqslant 1 \tag{4.12b}$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} = 0 \tag{4.13a}$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n)} = -\underset{\sim}{\nabla}\cdot\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n-1)} \quad , \quad n\geqslant 1 \tag{4.13b}$$

The simplest way of satisfying these equations is to assume

$$\underset{\sim}{\nabla}\cdot\underset{\approx}{U} \equiv 0 \tag{4.14}$$

$$\underset{\sim}{\nabla}\cdot\underset{\approx}{\beta} \equiv 0 \tag{4.15}$$

so that

$$i\underset{\sim}{k}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(n)} = 0 \quad , \quad \forall\, n \tag{4.16}$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n)} = 0 \quad , \quad \forall\, n \tag{4.17}$$

(4.14)-(4.17) are useful in the kinematic dynamo problem. However, in the hydromagnetic dynamo problem, the full equations (4.12)-(4.13) must be retained *(see section 6.4.4)*.

The solution to (2.13) may now be written

$$\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} = \frac{i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})}{i\omega+\eta k^2+i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})}\ \underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} \tag{4.18a}$$

$$\begin{aligned}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)} = \frac{1}{i\omega+\eta k^2+i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})}\Big\{ & i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})\ \underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)} + \overline{\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\ \underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} \\ & -(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{B}} + (\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)})\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{u}} \\ & -\overline{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} - \frac{\partial}{\partial t}(\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}) \\ & + 2i\eta\,\underset{\sim}{k}\cdot\underset{\sim}{\nabla}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} \Big\}\end{aligned} \tag{4.18b}$$

$$\begin{aligned}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n)} = \frac{1}{i\omega+\eta k^2+i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})}\Big\{ & i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})\ \underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(n)} + \overline{\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\ \underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(n-1)} \\ (n\geq 2)\qquad & -(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(n-1)})\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{B}} + (\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n-1)})\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{u}} \\ & -\overline{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n-1)} - \frac{\partial}{\partial t}(\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n-1)}) \\ & + 2i\eta\,\underset{\sim}{k}\cdot\underset{\sim}{\nabla}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} + \eta\nabla^2\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(n-2)} \Big\}\end{aligned} \tag{4.18c}$$

These expressions may now be used to obtain further expressions for $\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}$.

4.3.3 Calculation of $\overline{\underset{\sim}{u}'\times \underset{\approx}{B}'}$ in the first order smoothing approximation

The value of $\overline{\underset{\sim}{u}'\times \underset{\sim}{B}'_o}$ appropriate to the first order smoothing approximation may now be obtained in the form of a series. Making use of (4.11),

$$\overline{\underset{\sim}{u}'\times \underset{\sim}{B}'_o} = \mathcal{R}e\iiiint\limits_{\underset{\sim}{k}\omega\underset{\sim}{k}'\omega'} \overline{\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^*(\underset{\sim}{k},\omega)\times \underset{\approx}{\beta}\cdot d\underset{\sim}{Y}(\underset{\sim}{k}',\omega')}\, e^{i\{(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x}+(\omega'-\omega)t\}}$$

$$= \mathcal{R}e\iint\limits_{\underset{\sim}{k}\omega} \overline{\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^{(0)*}(\underset{\sim}{k},\omega)\times \underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}(\underset{\sim}{k},\omega)} \; +$$

$$+\; \mathcal{R}e\iint\limits_{\underset{\sim}{k}\omega}\Big\{ \overline{\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^{(0)*}(\underset{\sim}{k},\omega)\times \underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)}(\underset{\sim}{k},\omega)} \; +$$

$$+\; \overline{\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^{(1)*}(\underset{\sim}{k},\omega)\times \underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}(\underset{\sim}{k},\omega)}\Big\} \; +$$

$$+ \cdots \tag{4.19}$$

so that

$$\{\overline{\underset{\sim}{u}'\times \underset{\sim}{B}'_o}\}_i = \mathcal{R}e\Bigg[i\,\overline{B}_q \iint\limits_{\underset{\sim}{k}\omega} \frac{k_q\, d\underset{\sim}{k}\, d\omega}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})}\, \epsilon_{ij\ell}\, U^*_{jm}\,\cdot$$

$$\cdot\Big\{ U_{\ell p}\big[\phi^{(00)}_{mp}+\phi^{(10)}_{mp}+\phi^{(01)}_{mp}+\cdots\big] +$$

$$+\; U_{np}\frac{\partial \overline{u}_\ell}{\partial x_n}\cdot\frac{\phi^{(00)}_{mp}}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})}+\cdots\Big\} +$$

$$+\; \epsilon_{ij\ell}\iint\limits_{\underset{\sim}{k}\omega}\frac{d\underset{\sim}{k}\, d\omega}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})}\, U^*_{jm}\,\phi^{(00)}_{mp}\Big\{\overline{B}_n\frac{\partial}{\partial x_n}U_{\ell p} - U_{np}\frac{\partial}{\partial x_n}\overline{B}_\ell$$

$$-\Big[\overline{u}_n\frac{\partial}{\partial x_n}+\frac{\partial}{\partial t}-2i\eta k_n\frac{\partial}{\partial x_n}\Big]\frac{i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})\,U_{\ell p}}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})}\Big\} +$$

$$+\cdots\Bigg] \tag{4.19'}$$

4.3.4 Introduction of boundary conditions on the turbulent velocity - the possibility of a scalar velocity amplitude

The final step in the solution is to assume that U_{ij} , $\bar{\underset{\sim}{B}}$, and $\bar{\underset{\sim}{u}}$ are all functions of $(\underset{\sim}{x},t)$ in (4.19'). It will then be possible to satisfy any desired boundary conditions on $\underset{\sim}{u}'$ by adjusting the form of U_{ij} . It is of course clear that for the kinematic dynamo problem, the expansions (4.8a) and (4.10) are unnecessary. In addition, the complex tensor U_{ij} may be replaced with a real scalar *velocity amplitude* function $U(\underset{\sim}{x},t)$. Equation (4.19') then simplifies to

$$
\begin{aligned}
\{\overline{u'\times B_0'}\}_i = \mathrm{Re}\Bigg[\epsilon_{ij\ell}\,U(\underset{\sim}{x},t)\Big\{ & i\,U\bar{B}_q \iint_{\underset{\sim}{k}\omega} \frac{k_q\,\phi_{j\ell}(\underset{\sim}{k},\omega)}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})}\,d\underset{\sim}{k}\,d\omega \\
& + i\,U\bar{B}_q \frac{\partial \bar{u}_\ell}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{k_q\,\phi_{jn}}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})}\,d\underset{\sim}{k}\,d\omega \\
& + \bar{B}_n \frac{\partial U}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{\phi_{j\ell}\,d\underset{\sim}{k}\,d\omega}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})} - U\frac{\partial \bar{B}_\ell}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{\phi_{jn}\,d\underset{\sim}{k}\,d\omega}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})} \\
& - i\left(\bar{u}_n\frac{\partial}{\partial x_n}+\frac{\partial}{\partial t}\right)(U\bar{B}_q) \iint_{\underset{\sim}{k}\omega} \frac{k_q\,\phi_{j\ell}}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})^2}\,d\underset{\sim}{k}\,d\omega \\
& - U\bar{B}_q\left(\bar{u}_n\frac{\partial}{\partial x_n}+\frac{\partial}{\partial t}\right)\bar{u}_r \iint_{\underset{\sim}{k}\omega} \frac{k_q k_r\,\phi_{j\ell}}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})^3}\,d\underset{\sim}{k}\,d\omega \\
& - 2\eta\frac{\partial}{\partial x_n}(U\bar{B}_q) \iint_{\underset{\sim}{k}\omega} \frac{k_q k_n\,\phi_{j\ell}}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})^2}\,d\underset{\sim}{k}\,d\omega \\
& + 2i\eta\,U\bar{B}_q\frac{\partial \bar{u}_r}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{k_q k_n k_r\,\phi_{j\ell}}{(i\omega+\eta k^2+i\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})^3}\,d\underset{\sim}{k}\,d\omega + \cdots\Big\}\Bigg]
\end{aligned}
\tag{4.20}
$$

However, the more complicated expressions in (4.19') will be required later on, when we consider the hydromagnetic dynamo problem *(see section 6.4)*.

4.3.5 $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}$ for purely turbulent flow

When the mean velocity $\overline{\underset{\sim}{u}}$ and its gradient can be neglected, (4.20) reduces to

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}\}_i = \mathcal{R}e\Big[\epsilon_{ij\ell}\, U \Big\{ i U \overline{B}_q \iint_{\underset{\sim}{k}\omega} \frac{k_q \phi_{j\ell}}{(i\omega+\eta k^2)}\, d\underset{\sim}{k}\, d\omega$$

$$+ \overline{B}_n \frac{\partial U}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{\phi_{j\ell}}{(i\omega+\eta k^2)}\, d\underset{\sim}{k}\, d\omega \;-\; U \frac{\partial \overline{B}_\ell}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{\phi_{jn}}{(i\omega+\eta k^2)}\, d\underset{\sim}{k}\, d\omega$$

$$- i \frac{\partial}{\partial t}(U\overline{B}_q) \iint_{\underset{\sim}{k}\omega} \frac{k_q \phi_{j\ell}}{(i\omega+\eta k^2)^2}\, d\underset{\sim}{k}\, d\omega$$

$$- 2\eta \frac{\partial}{\partial x_n}(U\overline{B}_q) \iint_{\underset{\sim}{k}\omega} \frac{k_q k_n \phi_{j\ell}}{(i\omega+\eta k^2)}\, d\underset{\sim}{k}\, d\omega \;+ \ldots \Big\}\Big] \qquad (4.21)$$

If the turbulence is *locally PT-invariant* - i.e. if $\underset{\sim}{u}'$ can be represented in the form (4.7a), where $d\underset{\sim}{Z}$ satisfies (4.9), *and* ϕ_{ij} is the spectrum tensor of a PT-invariant process - then ϕ_{ij} is real, and

$$\phi_{ij}(k,\omega) = \phi_{ji}(k,\omega)$$

by (3.22). In this case, all but one of the terms in (4.21) drop out, and the equation reduces to

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'}\}_i = \mathcal{R}e\left\{-U^2 \epsilon_{ij\ell} \frac{\partial \overline{B}_\ell}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{\phi_{jn}}{(i\omega + \eta k^2)} d\underset{\sim}{k}\, d\omega\right\}$$

$$= -U^2 \epsilon_{ij\ell} \frac{\partial \overline{B}_\ell}{\partial x_n} \iint_{\underset{\sim}{k}\omega} \frac{\eta k^2 \phi_{jn}}{(\omega^2 + \eta^2 k^4)} d\underset{\sim}{k}\, d\omega \tag{4.22}$$

Equation (4.22) may be rewritten in the form

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}_o'} = -\eta \left(\underset{\approx}{\Lambda} \cdot \underset{\sim}{\nabla}\right) \times \underset{\sim}{\overline{B}} \tag{4.22'}$$

where the tensor Λ_{ij} is defined as

$$\Lambda_{ij} \equiv U^2 \iint \frac{k^2 \phi_{ij}}{(\omega^2 + \eta^2 k^4)} d\underset{\sim}{k}\, d\omega \tag{4.22''}$$

Since ϕ_{ij} is symmetric under interchange of indices, by the assumption of *PT-invariance*, Λ_{ij} must also be symmetric under interchange of indices. Furthermore, since the diagonal elements of ϕ_{ij} must be *separately* non-negative if *Bochner's Theorem* is to be satisfied *(Batchelor, 1953, p. 27)*, Λ_{ij} must also have non-negative diagonal elements.

For the particular case in which ϕ_{ij} is the spectrum tensor of a homogeneous, stationary, *isotropic* process, we have, from (3.7)

$$\phi_{ij} = \frac{\mathcal{E}(k,\omega)}{4\pi k^4} \left\{k^2 \delta_{ij} - k_i k_j\right\} \tag{4.23}$$

where $\varepsilon(k,\omega)$ is a non-negative function. Equations (4.22') and (4.22") then become

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'_o} = -\eta\Lambda \underset{\sim}{\nabla} \times \underset{\sim}{\overline{B}} = \frac{1}{\sigma}\Lambda \underset{\sim}{\overline{j}} \tag{4.24}$$

$$\Lambda_{ij} = \Lambda\,\delta_{ij} \tag{4.24'}$$

$$\Lambda \equiv \frac{2}{3} U^2 \frac{1}{\eta^2 \lambda_c^3 \tau_c} \int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\xi \, \frac{\xi^2 \mathcal{E}(\xi/\lambda_c, \nu/\tau_c)}{\xi^4 + (q\nu)^2} \tag{4.24"}$$

where λ_c and τ_c are to be interpreted as the correlation length and time, q is the parameter defined in (2.69), and ξ and ν are dimensionless. Substituting (4.24) into the *modified Ohm's Law*, (1.17), we obtain

$$\underset{\sim}{\overline{j}} = \sigma_T \underset{\sim}{\overline{E}} \tag{4.25}$$

where σ_T is the *turbulent conductivity*, defined by

$$\sigma_T = \sigma[1 + \Lambda]^{-1} \tag{4.26}$$

Since Λ is non-negative, the only effect of *locally isotropic turbulence* of the type considered here is to cause a *decrease* in the effective (turbulent) conductivity. This result is in agreement with the conclusion of *Sweet (1950)* that *isotropic turbulence must increase the rate of diffusive decay of a nearly uniform magnetic field.* *(See section 3.1.)*

The results of the last paragraph make it clear that if a scalar velocity amplitude U is assumed, the introduction of inhomogeneity and nonstationarity in the turbulence does nothing to remove the need for *asymmetry*

in the *local* spectrum tensor, ϕ_{ij} , if dynamo action is to be obtained.

4.3.6 Comparison of results with those of Chapter 3

The expression (4.26) for the turbulent conductivity can be compared with the expression obtained for the *effective diffusivity*, η_{eff} , in (3.91). Writing

$$\varepsilon(k,\omega) = C\cdot\lambda_c^5\tau_c\cdot k^4\cdot\hat{h}(k,\omega) \tag{4.27}$$

in agreement with (3.45) and (3.75), and rearranging terms in (4.24"), we obtain

$$(\eta_{eff}/\eta) - 1 = \frac{2}{3}C\left(\frac{U\lambda_c}{\eta}\right)^2\int_{-\infty}^{\infty}d\nu\int_0^{\infty}\frac{\xi^6}{\xi^4+(\nu q)^2}\hat{h}(\xi/\lambda_c, \nu/\tau_c)\,d\xi \tag{4.28}$$

Equation (4.28) is identical to (3.91), apart from the fact that U is now a function of position and time. This agreement provides a useful check on the validity of the successive approximation technique used in deriving (4.26).

4.3.7 Introduction of helicity through large-scale variations of the turbulent velocity

The correlation tensor used in the approach leading to equation (4.19) is of the form

$$R_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}+\underset{\sim}{r},t+\tau)$$

$$= \iiiint\limits_{\underset{\sim}{k}\omega\underset{\sim}{k}'\omega'} U^*_{i\ell}(\underset{\sim}{x},t;\underset{\sim}{k},\omega)\, U_{jm}(\underset{\sim}{x}+\underset{\sim}{r},t+\tau;\underset{\sim}{k}',\omega')\, \overline{dZ^*_\ell(\underset{\sim}{k},\omega)\, dZ_m(\underset{\sim}{k}',\omega')} \cdot$$

$$\cdot\, e^{i\{(\underset{\sim}{k}'-\underset{\sim}{k})\cdot\underset{\sim}{x} + (\omega'-\omega)t + \underset{\sim}{k}'\cdot\underset{\sim}{r} + \omega'\tau\}}$$

$$= \iint\limits_{\underset{\sim}{k}\omega} U^*_{i\ell}(\underset{\sim}{x},t;\underset{\sim}{k},\omega)\, U_{jm}(\underset{\sim}{x}+\underset{\sim}{r},t+\tau;\underset{\sim}{k},\omega)\, \phi_{\ell m}(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega\tau\}} d\underset{\sim}{k}\,d\omega \tag{4.29}$$

If the assumption concerning a *scalar velocity amplitude*, used in deriving (4.20), is made, (4.29) reduces to

$$R_{ij}(\underset{\sim}{x},t;\underset{\sim}{x}+\underset{\sim}{r},t+\tau) = U(\underset{\sim}{x},t)\,U(\underset{\sim}{x}+\underset{\sim}{r},t+\tau)\,\mathcal{M}_{ij}(\underset{\sim}{r},\tau) \tag{4.30}$$

where

$$\mathcal{M}_{ij}(\underset{\sim}{r},\tau) = \iint\limits_{\underset{\sim}{k}\omega} \phi_{ij}(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega\tau\}}\, d\underset{\sim}{k}\,d\omega \tag{4.31}$$

When (4.30) is used, the spectrum tensor is of the form

$$\Phi_{ij}(\underset{\sim}{k},\omega) = U^2\, \phi_{ij}(\underset{\sim}{k},\omega) \tag{4.32}$$

where the large-scale variations of U have been ignored in writing the arguments of Φ_{ij}. Since the symmetry properties of Φ_{ij} in (4.32) are just those of ϕ_{ij}, it is not to be expected, in this approximation, that large-scale variations in U will materially affect the symmetry requirements on Φ_{ij} for dynamo action to occur.

When (4.29) is used, on the other hand, the spectrum tensor is of the form

$$\Phi_{ij}(\underset{\sim}{k},\omega) = U^{*}_{il}(\underset{\sim}{k},\omega)\,U_{jm}(\underset{\sim}{k},\omega)\,\phi_{lm}(\underset{\sim}{k},\omega) \tag{4.33}$$

when the large-scale variations of U_{ij} are ignored. In (4.33) the symmetry properties of Φ_{ij} depend on both those of ϕ_{lm} and those of $U^{*}_{il}U_{jm}$, so that Φ_{ij} *may have an asymmetric component even when* ϕ_{ij} *does not.* Helicity may therefore arise from either small-scale or large-scale variations of $\underset{\sim}{u}'$. *(See section 6.4.6 for further discussion of this point.)*

4.3.8 Comparison of the successive approximation technique with *Rädler's* approach

We may now identify the principal difference between the successive approximation technique suggested in this thesis and the approach of *Rädler (1972)* and *Krause and Rädler (1971)*, described in *section 4.2.3*.

Krause and Rädler obtain helicity which is related to the large-scale variations of $\underset{\sim}{u}'$ by assuming R_{ij} to have the specific form (4.5). They then choose the coefficients $m^{(0)}_{ij}$, $m^{(11)}_{ij}$, and $m^{(12)}_{ij}$ to represent correlation tensors of stationary, homogeneous, isotropic processes, with the form (3.48), so that all the *helicity* effects arise from the large-scale variations. In the successive approximation technique used here, we introduce the effects of large-scale variations through the tensor

$U_{il}^{*}U_{jm}$ without making any *a priori* assumptions about the form of R_{ij}. The successive approximation technique allows $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ to be derived from the hydromagnetic dynamo equations in a much more general form than is possible with the method of *Krause and Rädler*. This topic will be studied in more detail in *section 6.4*.

4.4 Summary of Chapter 4

This chapter is concerned with nonstationary, inhomogeneous turbulence and its treatment within the framework of mean field electrodynamics.

A successive approximation technique is proposed, and is applied to the kinematic dynamo problem. Some of the results obtained are compared with expressions derived in *Chapter 3*.

The possibility of introducing helicity through large-scale variations of the turbulent velocity distribution is discussed, and the successive approximation technique is compared with the approach suggested by *Rädler (1972)*.

5. TEMPORAL BEHAVIOUR OF ASTROPHYSICAL MAGNETIC FIELDS

5.1 Introduction

5.1.1 Temporal variations of astrophysical magnetic fields

As was noted in *section 1.1.2*, many astrophysical magnetic fields vary in a complicated way with time. These variations are summarized for a number of fields in *Tables 13, 14, and 15.*

It will be seen that several of the temporal variations listed in *Tables 14 and 15* can be explained in terms of stellar or planetary rotation. If the magnetic field of a rotating body is not symmetric about the axis of rotation, the field must vary periodically when viewed from the Earth. The simplest possible case of this type is that in which the magnetic field is predominantly dipolar, and the magnetic dipole axis is inclined to the axis of rotation. This *oblique rotator model* is clearly applicable to planetary fields like those of the Earth and Jupiter. It has also been applied with considerable success to stellar magnetic fields *(see, for example, Mestel, 1967, 1971, 1972; Preston, 1967a,b, 1971a,b; Landstreet, 1970; Mestel and Takhar, 1972).*

The *oblique rotator model* does not, however, provide a unique explanation for the periodic variations of stellar magnetic fields. An alternative model has been proposed by

Krause (1971, 1972b,c), in which the stellar field is assumed to be *nondipolar*, and symmetric under reflection in the equatorial plane. This *equator-symmetric rotator model* takes into account the fact that in some dynamo models including an α-effect *(Stix, 1971; Roberts and Stix, 1972; Krause, 1972c; Moffatt, 1973)* non-axisymmetric mean fields are more easily excited than axisymmetric mean fields.

TABLE 13 - GEOMAGNETIC FIELD OF INTERNAL ORIGIN: TEMPORAL VARIATION

Type of variation	Time scale (years)	Comments
SECULAR VARIATION OF NONDIPOLE FIELD		
"High frequency" oscillations [1]	< 100	Complicated spectrum of variations
"Medium frequency" oscillations [2]	100-5000	
"Westward drift" [3]	∿ 2000	
CHANGES IN ENERGY DISTRIBUTION		In the last 120y the net energy loss rate has been ∿0.02%/y, and the dipole-nondipole transfer rate ∿0.06%/y (percent of total field). However, over the last 1500y, the average loss rate has been much higher (∿0.13%/y). [5]
Dipole-nondipole energy transfer [4]	∿ 2000	
Changes in transfer rate [5]	∿ 100	
Changes in total energy [5]	10^3-10^4	
SECULAR VARIATION OF DIPOLE FIELD		
Dipole "wobble" (eastward drift) [6]	1200-1800	Various estimates
Field strength oscillation [7]	∿ 9000	"Fundamental frequency"
POLARITY REVERSALS		
Change in field direction [8,9,10]	1000-4000	[8]: intensity time scale ∼10x direction time scale
Change in intensity [8,9]	3000-10,000	[9]: both time scales the same (3500y in two cases)
Interval between reversals [11]	0.03-30 x 10^6	2-3 x 10^5y over the last 50 my.
Long-term periodicity(?) [12]	75 x 10^6	Solar vibration ⊥ galactic plane??
	250 x 10^6	Galactic rotation???
	700 x 10^6	??

FOOTNOTES TO TABLE 13

[1] Braginskii (1964d, 1970a,b, 1971, 1972); Currie (1968); Acheson & Hide (1973).

[2] Braginskii (1964b, 1967b, 1970b, 1971, 1972); Hide (1966a); Rikitake (1966b); Malkus (1967b, 1971a); Stewartson (1967, 1971); Gans (1971); Hide & Stewartson (1972); Soward (1972a); Roberts & Soward (1972); Acheson & Hide (1973).

[3] *For early references see* Jacobs (1963, pp. 70-76), Rikitake (1966a, p. 83 and p. 109). *See also* Pudovkin & Valuyeva (1967, 1972); Yukutake (1968a,b, 1972); James (1968, 1970, 1971); Honkura & Rikitake (1972); Roberts & Soward (1972); Moffatt (1973).

[4] McDonald & Gunst (1968); Verosub & Cox (1971); Cox (1972); Jin (1973).

[5] Verosub & Cox (1971).

[6] Kawai & Hirooka (1967); Kovacheva (1969); Márton (1970); Cox (1972). *See also* Jacobs (1971c); Pudovkin & Valuyeva (1972).

[7] Smith (1967); Braginskii (1970b, 1971, 1972); Cox (1972).

[8] Dunn, et al. (1971).

[9] Kent, et al. (1973).

[10] Harrison & Somayajulu (1966); Bullard (1968); Creer & Ispir (1970); Cox (1972).

[11] Bullard (1968); Heirtzler, et al. (1968); Helsley & Steiner (1969); McElhinny (1971); Vogt, et al. (1972); Helsley (1972a); Blakely & Cox (1972b); Stewart & Irving (1973); Reid (1973).

[12] Crain, et al. (1969); Crain & Crain (1970); Ulrych (1972).

TABLE 14 - TEMPORAL VARIATIONS OF SOLAR MAGNETIC FIELDS

Type of variation	Time scale	Comments
FILAMENTARY STRUCTURE VARIATIONS		
Granular structure [1]	10–15m	Variations associated with velocity fields in solar atmosphere.
Supergranular structure [2]	20–24h	
Giant cell structure [3]	∿ 30d	
DIFFERENTIAL ROTATION		Active photospheric field rotates 1°/day faster than solar atmosphere [4], while sector structure rotates with atmosphere [5]. Atmospheric rotation period at equator is ∿27d , while that at the poles is ∿34d [4].
Equatorial active regions [4]	∿ 25d	
Sector structure [5]	∿ 27d	
SOLAR CYCLE VARIATIONS		
Latitude migration of zones of maximum activity [6]	11–13y	Although the generally accepted solar cycle period is ∿11y, individual "cycles" of activity overlap. The length of each cycle is 13y [9] Magnetic cycle period ∿22y.
Reversal of characteristic polarity of spot zones [7]		
Reversal of axial dipole field [8]		There is some question as to the regularity of dipole reversals [10]
80-YEAR CYCLE VARIATIONS		
Sunspot activity [11]	∿ 80y	
Sector structure activity [12]	80–100y	Periodicity has not been definitely established.

FOOTNOTES TO TABLE 14

[1] Parker (1970b); Mehltretter (1971); Harvey (1971); Weiss (1971a, 1972).

[2] Parker (1970b); Howard (1971a,b); Weiss (1971a, 1972).

[3] Howard (1971a,b); Weiss (1971a, 1972).

[4] Dupree & Henze (1972).

[5] Severny, et al. (1970); Wilcox & Gonzales (1971); Svalgaard (1973).

[6] Parker (1970b); Stenflo (1972).

[7] Stenflo (1972).

[8] Severny (1971, 1972); Stenflo (1972).

[9] de Jager (1959); *see also* Gilliland (1967, p. 159).

[10] Stenflo (1972).

[11] Kopecky (1970).

[12] Patterson (1973).

TABLE 15 - TEMPORAL VARIATIONS OF OTHER ASTROPHYSICAL MAGNETIC FIELDS

Location of field	Time scale of variation	Nature and explanation of variation
Jupiter	9h 55m	Periodic variation. Explained in terms of the *oblique rotator model*, with dipole axis inclined 11° to axis of rotation. [1]
Magnetic stars	1.7-2500 d *typically 5-9 d*	All vary, and many show periodic variation. About 50% exhibit *polarity reversals*. [2] Most features can be explained in terms of stellar rotation. On the *oblique rotator model* [3], the majority of magnetic stars have their dipole axis inclined ∿80° to rotation axis, though some have inclinations ∿0° [4]. Alternatively, the *equator-symmetric rotator model* [5] may apply. Fluctuations and certain other features may be associated with *wave processes* [6].
Magnetic white dwarfs	(1.34d)	Only one of the four magnetic white dwarfs observed to date exhibits periodic variation [7]. Variation is explained in terms of the *oblique rotator model*.
Pulsars	0.03-3.7 s *most cases, 0.5-1 s*	All vary in a rapid periodic fashion. Periodic variations are explained in terms of the *oblique rotator model* [8]. *Glitches* (sudden jumps in frequency) are also observed, with "lifetimes" *4-7 d* (Crab pulsar) and *1.2 y* (Vela). Several explanations have been proposed [9].

FOOTNOTES TO TABLE 15

[1] Warwick (1967); Hide (1971a); Schatten & Ness (1971).

[2] Ledoux & Renson (1966); Preston (1967a,b, 1971b).

[3] Mestel (1967, 1971, 1972); Mestel & Takhar (1972); Moffatt (1973).

[4] Preston (1967b, 1971b); Landstreet (1970).

[5] Krause (1971).

[6] Severny (1971).

[7] Landstreet & Angel (1971).

[8] Ruderman (1972).

[9] Ruderman (1972).

5.1.2 Polarity reversals

As may be seen from *Tables 13-15*, the magnetic fields of the Earth, the Sun, and some magnetic stars all exhibit *polarity reversals*. In the case of magnetic stars, these reversals are readily explained in terms of either the *oblique rotator model* or the *equator-symmetric rotator model*. However, these models cannot be applied to the magnetic fields of the Earth and the Sun.

Reversals of the solar and geomagnetic fields differ considerably in character. The background solar field appears to reverse in a quasi-periodic fashion related to the 22-year *magnetic solar cycle*. The change-over from one polarity to the other is not always smooth - for example, the field is sometimes *quadrupolar* rather than dipolar during a reversal. [Recent observations *(Stenflo, 1972)* suggest that irregularities of this nature are more common and of longer duration than would be expected on the simple "periodic reversal" model.] The geodynamo, on the other hand, reverses in a highly irregular fashion - so irregular that geomagnetic reversals are often modelled as a *nonstationary random process (Cox, 1968, 1969, 1970, 1971; Surdin, 1968; Nagata, 1969; Crain and Crain, 1970; Crain, 1971; Naidu, 1971; Kono, 1972; Blakely and Cox, 1972a; Phillips, et al., 1972).*

The frequency of geomagnetic reversals is plotted as a function of time in *Figure 22*. The smooth, dashed curve has been obtained from the work of *McElhinny (1971)*, who plots the percentage of polarity measurements that are "mixed" *(i.e. both "normal" and "reversed" polarities in the same rock unit)* as a function of time. As shown in *Figure 22*, the most recent portions of *McElhinny's* curve can be fitted quite well to detailed reversal frequency measurements if the curve is considered as a *logarithmic plot of frequency vs. time*. The curve is clearly a very crude approximation, but it serves to illustrate the non-stationary character of the reversal process.

The detailed results of *Heirtzler, et al. (1968)*, and of *Helsley and Steiner (1969)*, plotted on the right in *Figure 22*, indicate that changes in the reversal frequency are likely to be *discontinuous* on the time scale shown. There appear to be sudden jumps in the frequency of reversals at 50 m.y.b.p. *(million years before present)*, and at 72 m.y.b.p.

Other results, not plotted in *Figure 22*, indicate that the nonstationary character of the reversal process has persisted over much longer times than those indicated. *Reid (1972)* and *Stewart and Irving (1973)* have found that reversal rates in the Precambrian varied in much the same way as those plotted for the Phanerozoic *(i.e. the entire*

period shown in Figure 22). Reid (1972) reports a variation of the reversal rate from 0.4/m.y. to 1.1/m.y. over a 60 m.y. interval at roughly 1800 m.y.b.p. *Stewart and Irving (1973)* report reversal frequencies less than 0.1/m.y. at 990 m.y.b.p., and greater than 1/m.y. at 790 m.y.b.p.

Since *Figure 22* was plotted, two papers have been discovered which indicate that the peak shown in the Jurassic and Triassic is somewhat too low. *Vogt, Einwich, and Johnson (1972)* report 41 reversals between 150 and 135 m.y.b.p., giving an average reversal rate of 2.7/m.y. in the late Jurassic. *Helsley (1972a)* reports that at least 23 reversals occurred during the Triassic, giving an average reversal rate > 0.7/m.y. between 225 and 190 m.y.b.p.

Detailed study of more recent palaeomagnetic data indicates that the time between reversals varies widely - from $\sim 3 \times 10^4$ years to as long as 3×10^7 years *(Bullard, 1968; Heirtzler, et al., 1968; Blakely and Cox, 1972b; Moffatt, 1973)*. During the last few million years, reversals have occurred at intervals of roughly $2\text{-}3 \times 10^5$ years.

The behaviour of the geomagnetic field *during* a polarity reversal has received considerable attention in recent years. *Dunn, et al. (1971)*, in a study of a reversal at 15 m.y.b.p., find that the field intensity decreased by a factor of 10 before any change in field direction occurred, and did not return to normal until after the directional change was completed. The directional

Figure 22. Reversal rate of the geomagnetic field as a function of time.

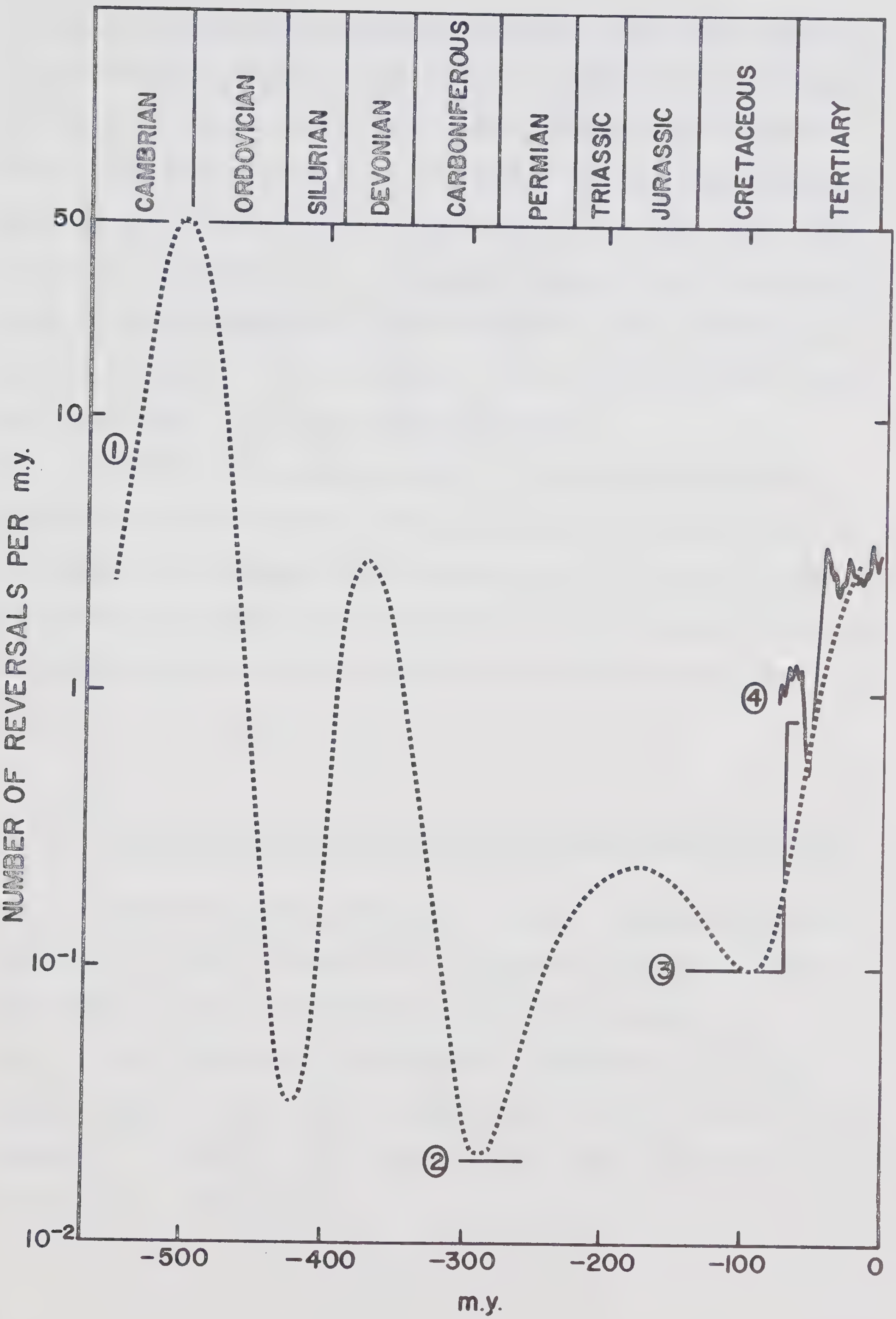
(1) Curve derived from the work of *McElhinny (1971)*. McElhinny's plot of percentage of "mixed" polarity measurements vs. time is interpreted, to a crude approximation, as a *logarithmic* plot of reversal rate vs. time.

(2) Kiaman Magnetic Interval - approximately one reversal in 50 m.y.

(3) Results of *Helsley and Steiner (1969)*.

(4) Results of *Heirtzler, et al. (1968)*, obtained from sea-floor anomaly patterns. (Plot shows a 10 m.y. average taken every 1 m.y.)

As pointed out on p. 278, the work of *Vogt, Einwich, and Johnson (1972)* and of *Helsley (1972a)* indicates that the peak shown in the Jurassic and Triassic is perhaps an order of magnitude too low.


CAMBRIAN
ORDOVICIAN
SILURIAN
DEVONIAN
CARBONIFEROUS
PERMIAN
TRIASSIC
JURASSIC
CRETACEOUS
TERTIARY
NUMBER OF REVERSALS PER m.y.
50
10
1
10⁻¹
10⁻²
-500
-400
-300
-200
-100
0
m.y.
①
②
③
④

change is estimated to have taken 1-4 x 10^3 years, while the intensity change is estimated to have taken 10^4 years. Similarly, *Watson and Larson (1972)* suggest that "major dipole instabilities" (e.g. reduction of the dipole intensity by 80%) occur prior to reversals. On the other hand, *Kent, et al. (1973)*, in a detailed study of two reversals, find that the change in field direction took the same time as the change in field intensity in each case (3800 y for one reversal, and 3500 y for the other).

Recent work suggests that "...the field during a [polarity] transition is *not* a geocentric dipole tightly coupled to the mantle" *(Hillhouse, et al., 1972)*. Large variations in magnetic inclination and declination, as well as a decrease in field intensity, occur during a reversal *(Kent, et al., 1973)*.

5.1.3 The oscillation spectrum of the geomagnetic field

As may be seen from *Table 13*, the geomagnetic field varies with time on many scales besides that of polarity reversals. It is frequently helpful to consider an *oscillation spectrum* of geomagnetic variations *(Cox and Doell, 1964; Jacobs, 1970b; Braginskii, 1970b, 1971, 1972)*. *Braginskii (1970b, 1971, 1972)* divides this "spectrum" into three major categories:

a. the *fundamental frequency*, characteristic of dipole field strength oscillations, with period $\sim 9 \times 10^3$ years *(see footnote* [7], *Table 13)*;

b. *medium frequency oscillations*, with periods in the range 100-5000 years (typically $\sim 10^3$ years, with peaks in the spectrum close to the period of *westward drift* of the nondipole field);

c. *high frequency oscillations*, with periods < 100 years *(see footnote* [1], *Table 13)*.

The theoretical treatment of these oscillations is highly complicated. *Braginskii (1970b, 1971, 1972)* suggests that the existence of a *fundamental frequency* is a consequence of the *two-stage* nature of the dynamo process, in which a weak poloidal field leads to the generation of a strong toroidal field, and the toroidal field is responsible for the regeneration of the poloidal field. *Medium frequency oscillations* are associated with the so-called "MAC-waves" *(i.e. Magnetic-Archimedean-Coriolis waves)* in the Earth's fluid core *(see the references in footnote* [2], *Table 13)*. Finally, *high frequency oscillations* are linked to torsional magnetohydrodynamic oscillations and turbulent pulsations in the core *(see the references in footnote* [1], *Table 13)*.

5.2 Reversals and the mean field kinematic dynamo problem

5.2.1 Dynamo models for solar and geomagnetic reversals

In the last few years, a number of reasonably successful dynamo models have been developed to account for the temporal variation of the solar magnetic field *(Gilman, 1968, 1969a,b; Leighton, 1969; Steenbeck and Krause, 1969a; Parker, 1970b; Deinzer and Stix, 1971; Stix, 1971; Roberts and Stix, 1972; Stix, 1972; Lerche and Parker, 1972)*. However, the problem of the long-term temporal variation of the geomagnetic field has proved much less tractable. Some progress has been made by *Parker (1969b)* and by *Levy (1972a,b,c)*, who have carried out kinematic investigations of geomagnetic reversals. The model suggested by *Levy (1972a,b,c)* is of particular interest because of its relative mathematical simplicity. However, like most kinematic models, it suffers from the disadvantage that its velocity distribution may well be a very poor representation of the distribution actually present in the Earth's fluid core.

5.2.2 The mean field induction equation

Let us first consider the *mean field kinematic dynamo problem* as it applies to the Earth. It is convenient to separate the magnetic field into its *toroidal* and *poloidal* parts

$$\overline{\underset{\sim}{B}} = \text{curl}\, T\underset{\sim}{r} + \text{curl}\,\text{curl}\, S\underset{\sim}{r} \tag{5.1}$$

where $\underset{\sim}{r}$ is the position vector, and $T(\underset{\sim}{r},t)$ and $S(\underset{\sim}{r},t)$ are scalar fields *(see the footnote to Table 9, p. 33, and P.H. Roberts, 1967a, pp. 80-82).*

Using spherical polar coordinates $[r,\theta,\phi]$, and assuming that T and S are functions only of $[r,\theta,t]$, we may rewrite (5.1) in the form

$$\begin{aligned} \overline{\underset{\sim}{B}} &= -\frac{\partial T}{\partial \theta}\underset{\sim}{1}_\phi - \text{curl}\left\{\frac{\partial S}{\partial \theta}\underset{\sim}{1}_\phi\right\} \\ &\equiv B\,\underset{\sim}{1}_\phi + \text{curl}\left\{A\underset{\sim}{1}_\phi\right\} \end{aligned} \tag{5.2}$$

where $\underset{\sim}{1}_\phi$ is the unit vector in the azimuthal direction. We shall also assume that the mean field induction equation has the form

$$\left\{\frac{\partial}{\partial t} - \eta\nabla^2\right\}\overline{\underset{\sim}{B}} = \text{curl}\left\{\overline{\underset{\sim}{u}} \times \overline{\underset{\sim}{B}} + \alpha\overline{\underset{\sim}{B}}\right\} \tag{5.3}$$

where an α-*effect* term $\alpha\overline{\underset{\sim}{B}}$ has been substituted for the fluctuating e.m.f. $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ *(see the discussion in section 1.4.8).*

5.2.3 The α^2 dynamo

If, in (5.3), we make the further assumptions that $\bar{\underset{\sim}{u}}$ and α are functions only of $[r,\theta,t]$, and that

$$\bar{\underset{\sim}{u}} = u \underset{\sim}{1}_\phi + \text{curl}\{a \underset{\sim}{1}_\phi\} \tag{5.4}$$

equations (5.2), (5.3), and (5.4) can be combined to give

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} A = \tag{5.5a}$$

$$= \alpha B + \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}(ra)\frac{\partial}{\partial\theta}(A\sin\theta) + \frac{\partial}{\partial\theta}(a\sin\theta)\frac{\partial}{\partial r}(rA)\right\}$$

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} B = \tag{5.5b}$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}\left\{\alpha\frac{\partial}{\partial r}(rA)\right\} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left\{\alpha\frac{\partial}{\partial\theta}(A\sin\theta)\right\}$$

$$+ \left\{\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{\cot\theta}{r}\frac{\partial}{\partial r}\right\}(aB - uA)$$

$$+ \frac{1}{r}\left\{\frac{\partial u}{\partial r}\frac{\partial A}{\partial\theta} - \frac{\partial u}{\partial\theta}\frac{\partial A}{\partial r} - \frac{\partial B}{\partial r}\frac{\partial a}{\partial\theta} + \frac{\partial B}{\partial\theta}\frac{\partial a}{\partial r}\right\}$$

where

$$\Delta_1 \equiv \left\{\text{div}\cdot\text{grad} - \frac{1}{r^2\sin^2\theta}\right\} = \left\{\nabla^2 - \frac{1}{r^2\sin^2\theta}\right\} \tag{5.6}$$

If $\bar{\underset{\sim}{u}} = 0$, equations (5.5a,b) reduce to

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} A = \alpha B \tag{5.7a}$$

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} B = -\frac{1}{r}\left\{\alpha\frac{\partial}{\partial r}(rA)\right\} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left\{\alpha\frac{\partial}{\partial\theta}(A\sin\theta)\right\} \tag{5.7b}$$

giving the equations of the so-called α^2 *dynamo (Krause and Steenbeck, 1967; P.H. Roberts, 1971a; Soward, 1972a; Roberts and Stix, 1972).*

Equation (5.7a) represents the generation of poloidal field (represented by A) from toroidal field (represented by B), while equation (5.7b) represents the generation of toroidal field from poloidal. The name "α^2 dynamo" derives from the fact that in the *two-stage* process described by equations (5.7a,b) the operation of each stage depends on the fact that $\alpha \neq 0$.

5.2.4 The $\alpha\omega'$ dynamo

If, in equation (5.4) $a = 0$, and if the terms involving α in equation (5.5b) can be ignored in comparison with the terms involving u, equations (5.5a,b) reduce to

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} A = \alpha B \tag{5.8a}$$

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} B = \frac{\cot\theta}{r}\frac{\partial}{\partial r}(uA) - \frac{1}{r^2}\frac{\partial}{\partial\theta}(uA) + \frac{1}{r}\left\{\frac{\partial u}{\partial r}\frac{\partial A}{\partial\theta} - \frac{\partial u}{\partial\theta}\frac{\partial A}{\partial r}\right\} \tag{5.8b}$$

giving the equations of the so-called $\alpha\omega'$ *dynamo (Steenbeck and Krause, 1966; Soward, 1972a).* The dynamo equations (5.8a,b), and generalizations of them, have been

studied by many authors, including *Parker (1955, 1970a,c, 1971c), Braginskii (1964a,b,c), Krause and Steenbeck (1965), Steenbeck and Krause (1967, 1969a,b), Krause and Rädler (1971)*, and *Roberts and Stix (1972)*.

u is often assumed to have the form of a "modified" rigid body rotation

$$u = \omega r \sin\theta$$

where $\omega = \omega(r,\theta)$. Under this assumption, equations (5.8a,b) reduce to

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} A = \alpha B \tag{5.8a'}$$

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} B = \underset{\sim}{1}_\phi \cdot \left\{\underset{\sim}{\nabla}\omega \times \underset{\sim}{\nabla}(Ar\sin\theta)\right\} \tag{5.8b'}$$

If, in addition, ω is independent of θ , the equations reduce further to

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} A = \alpha B \tag{5.8a''}$$

$$\left\{\frac{\partial}{\partial t} - \eta\Delta_1\right\} B = \omega' \frac{\partial}{\partial\theta}(A\sin\theta) \tag{5.8b''}$$

where $\omega' \equiv \omega'(r) = d\omega/dr$.

The reason for the name "$\alpha\omega'$ dynamo" is clear from equations (5.8a") and (5.8b"). In the two-stage process described by the equations, the first stage (generation of poloidal field from toroidal) operates only if $\alpha \neq 0$, while the second stage (generation of toroidal field from

poloidal) operates only if $\omega' \neq 0$.

5.2.5 The Levy model for geomagnetic reversals

The model studied by *Levy (1972a,b,c)* is of the $\alpha\omega'$ type, with

$$u = \{\omega r \sin\theta\} H(r_c - r) \tag{5.9}$$

where $H(x)$ is the *Heaviside unit function*, equal to unity for $x > 0$ and zero for $x < 0$. *Levy (1972a,b)*, like *Parker (1969b)*, represents α as a *pair of axisymmetric δ-function rings*, symmetrically placed above and below the equatorial plane. This model can be taken to represent two rings of cells of *cyclonic convection*.

Levy (1972b) studies field reversals by using a *sporadic* model in which "bursts" of shear (i.e. δ-functions in time) of the form (5.9) alternate with bursts of cyclonic convection at various latitudes. Two kinematic reversal schemes are discussed. In the first, it is shown that "...a strong burst of cyclonic convection at high latitudes will reverse the dipole field", while in the second it is shown that "...if the geomagnetic dynamo has a region of *reverse toroidal flux* in the core, then a strong burst of cyclonic convection in that region will also reverse the dipole field" *(Levy, 1972c)*.

Levy (1972c) examines the second kinematic reversal

scheme in more detail. He considers a model in which α represents *two or three pairs of rings of cyclonic convection*, and shows that stationary solutions of the dynamo equations exist in which extensive regions of reverse toroidal flux occur. It is estimated that in these solutions, a fluctuation of 20-30% in the distribution of cells of cyclonic convection can lead to a polarity reversal.

5.3 The $\alpha^2(r)$ dynamo in a spherical shell

5.3.1 The mean field induction equation and boundary conditions

In order to test the possibility of an α^2 dynamo model for geomagnetic reversals, let us consider equations (5.6a,b) in the case when α depends only on r. Under this assumption, the equations reduce to

$$\left\{\frac{\partial}{\partial t} - \eta\nabla^2\right\} S = \alpha T \tag{5.10a}$$

$$\left\{\frac{\partial}{\partial t} - \eta\nabla^2\right\} T = -\alpha\nabla^2 S - \frac{\alpha'}{r}\frac{\partial}{\partial r}(rS) \tag{5.10b}$$

where T and S are the scalar fields defined in (5.1) and (5.2), and $\alpha' = \alpha'(r) = d\alpha/dr$. Equations (5.10a,b) will be assumed valid in the *spherical shell* $r^* \leqslant r \leqslant r_o$, bounded by a nonconducting medium in $r > r_o$. In $r < r^*$ the mean velocity $\bar{\underset{\sim}{u}}$ may be nonzero, and equations (5.8a,b) may hold in this region in place of equations (5.10a,b).

The boundary conditions to be satisfied by the mean magnetic field are

$$\langle S\rangle = \langle T\rangle = \langle \partial S/\partial r\rangle = 0, \quad r = r_o \tag{5.11}$$

$$\langle S\rangle = \langle T\rangle = \langle \partial S/\partial r\rangle = \langle \partial T/\partial r\rangle = 0, \quad r = r^* \tag{5.12}$$

$$\begin{cases} S \to 0 & \text{as } r \to \infty \\ T = 0, & r \geqslant r_o \end{cases} \tag{5.13}$$

$$\begin{cases} S \text{ nonsingular as } r \to 0 \\ T \to 0 \text{ as } r \to 0 \end{cases} \tag{5.14}$$

The velocity field will be assumed to satisfy the boundary conditions

$$\alpha = 0 \ , \quad r \geq r_o \tag{5.15a}$$

$$\langle\alpha\rangle = \langle\alpha'\rangle = \langle\bar{\underset{\sim}{u}}\rangle = \langle\partial\bar{\underset{\sim}{u}}/\partial r\rangle = 0 \ , \quad r = r^* \tag{5.15b}$$

5.3.2 Separable solutions - the radial equations

In the region $r^* \leq r \leq r_o$ the equations (5.10a,b) have separable solutions of the form

$$S = \sum_{n=1}^{\infty} S_n(r)\, P_n(\cos\theta)\, e^{-\sigma_n t} \tag{5.16a}$$

$$T = \sum_{n=1}^{\infty} T_n(r)\, P_n(\cos\theta)\, e^{-\sigma_n t} \tag{5.16b}$$

where the *radial functions* $S_n(r)$ and $T_n(r)$ satisfy the equations

$$\left\{\sigma_n - \eta\left[\frac{d^2}{dr^2} - \frac{n(n+1)}{r^2}\right]\right\}(r S_n) = \alpha \cdot (r T_n) \tag{5.17a}$$

$$\left\{\sigma_n - \eta\left[\frac{d^2}{dr^2} - \frac{n(n+1)}{r^2}\right]\right\}(r T_n) = \tag{5.17b}$$

$$= -\alpha\left[\frac{d^2}{dr^2} - \frac{n(n+1)}{r^2}\right](r S_n) - \alpha' \frac{d}{dr}(r S_n)$$

and the $P_n(\cos\theta)$ are Legendre polynomials in $\cos\theta$.

In the region $r > r_o$ the conditions (5.13) lead

to solutions of the form

$$\hat{S}_n(r) = C_n r^{-(n+1)} \tag{5.18a}$$

$$\hat{T}_n(r) = 0 \tag{5.18b}$$

5.3.3 <u>The radial equations in nondimensional form</u>

In order to reduce the equations (5.17a,b) to a nondimensional form, we shall define

$$z \equiv 1 - (r/r_0) \tag{5.19}$$

$$X \equiv (rS_n)/r_0 S_n(r_0) \tag{5.20a}$$

$$Y \equiv (rT_n)/S_n(r_0) \tag{5.20b}$$

$$\gamma \equiv \sigma_n r_0^2/\eta \tag{5.21}$$

$$R \equiv r_0 \alpha(r)/\eta \tag{5.22}$$

We shall also use the notation

$$N \equiv n(n+1) \tag{5.23}$$

Substituting (5.19)-(5.23) into equations (5.17a,b), we obtain the nondimensional equations

$$\left\{ \frac{d^2}{dz^2} - \left[\frac{N}{(1-z)^2} + \gamma \right] \right\} X = -RY \tag{5.24a}$$

$$\left\{ \frac{d^2}{dz^2} - \left[\frac{N}{(1-z)^2} + \gamma \right] \right\} Y = \tag{5.24b}$$

$$= R \left\{ \frac{d^2}{dz^2} - \frac{N}{(1-z)^2} \right\} X + \frac{dR}{dz} \frac{dX}{dz}$$

These equations are valid in the interval $0 \leqslant z \leqslant z^*$, where

$$z^* \equiv 1 - (r^*/r_o) \tag{5.25}$$

5.3.4 Power series solutions of the radial equations

Since the problem is kinematic, the function $R(z)$ may be specified in any way which gives an *allowable flow*. We shall assume that $R(z)$ has a power series representation in $z \leqslant z^*$.

$$R(z) = \sum_{m=0}^{\infty} R_m z^m \tag{5.26}$$

The solutions to (5.24a,b) may also be expressed as power series in $z \leqslant z^*$.

$$X(z) = \sum_{m=0}^{\infty} X_m z^m \tag{5.27a}$$

$$Y(z) = \sum_{m=0}^{\infty} Y_m z^m \tag{5.27b}$$

Substituting (5.26) and (5.27) into (5.24), and

equating coefficients of powers of z, we find that

$$(k+2)(k+1)X_{k+2} - N\sum_{\ell=0}^{k}(k-\ell+1)X_\ell - \gamma X_k = \tag{5.28a}$$

$$= -\sum_{\ell=0}^{k} R_{k-\ell}Y_\ell$$

$$(k+2)(k+1)Y_{k+2} - N\sum_{\ell=0}^{k}(k-\ell+1)Y_\ell - \gamma Y_k = \tag{5.28b}$$

$$= \sum_{\ell=0}^{k}(\ell+2)(\ell+1)R_{k-\ell}X_{\ell+2} - N\sum_{\ell=0}^{k}(k-\ell+1)\sum_{t=0}^{\ell}R_{\ell-t}X_t$$

$$+ \sum_{\ell=0}^{k}(k-\ell+1)(\ell+1)R_{k-\ell+1}X_{\ell+1}$$

The mean field and velocity boundary conditions at $z = 0$ give

$$X_0 = 1 \quad , \quad X_1 = n \tag{5.29}$$

$$Y_0 = 0 \tag{5.30}$$

$$R_0 = 0 \tag{5.31}$$

Thus, from (5.28a,b),

$$X_2 = \tfrac{1}{2}(N+\gamma) \tag{5.32}$$

$$X_3 = \tfrac{1}{6}\{N(n+2) + \gamma\} \tag{5.33}$$

$$X_4 = \tfrac{1}{12}\{N(3+2n) + \tfrac{1}{2}(N+\gamma)^2 - R_1Y_1\} \tag{5.34}$$

$$X_5 = \tfrac{1}{20}\Big\{N(4+3n) + N(N+\gamma)\Big(1+\frac{n+2}{6}\Big) + \tfrac{1}{6}\gamma(N+\gamma)$$
$$-R_1Y_2 - R_2Y_1\Big\} \tag{5.35}$$

and

$$Y_2 = \frac{n}{2} R_1 \tag{5.36}$$

$$Y_3 = \frac{1}{6}\left\{(N+\gamma)Y_1 + (N+2\gamma)R_1 + 2nR_2\right\} \tag{5.37}$$

$$Y_4 = \frac{1}{12}\left\{ 2NY_1 + \left[\frac{n}{2}(\gamma-N) - 2N + \frac{3}{2}N(n+2) + \frac{3}{2}\gamma\right]R_1 + \right.$$
$$\left. + (2N+3\gamma)R_2 + 3nR_3 \right\} \tag{5.38}$$

$$Y_5 = \frac{1}{20}\left\{ \left[\frac{1}{6}N^2 + 3N - \frac{4}{3}R_1^2\right] + \left[\frac{5}{3}nN + \frac{1}{3}N^2 + N\right]R_1 + \right.$$
$$+ \left[\frac{4}{3}nN + 2N\right]R_2 + 3NR_3 + 4NR_4 +$$
$$+ \gamma\left[\frac{1}{3}NY_1 + \frac{4}{3}NR_1 + (2+\frac{1}{3}n)R_2 + 4R_3\right] +$$
$$\left. + \gamma^2\left[\frac{1}{6}Y_1 + R_1\right]\right\} \tag{5.39}$$

The parameters Y_1 and γ remain to be determined.

5.3.5 Application of the boundary conditions - the kinematic approach

From (5.12), there are four boundary conditions on the mean magnetic field to be satisfied at $z = z^*$ - namely,

$$\langle X\rangle = \langle Y\rangle = \langle dX/dz\rangle = \langle dY/dz\rangle = 0 \tag{5.40}$$

In the normal approach to the solution of the problem, these conditions are satisfied by adjusting Y_1 , γ , and the two arbitrary constants available in the solution of the induction equation in $z^* \leq z \leq 1$ after the conditions (5.14) have been satisfied.

There is, however, an alternative approach which may be used. Since the velocity field in the kinematic dynamo problem is arbitrary, apart from the requirements that it be *allowable* and that it satisfy the conditions (5.15), we may choose to fulfil the boundary conditions at $z = z^*$ *by adjusting the coefficients* R_n. In this approach we may assign any desired value to Y_1. For example, we may plausibly require that

$$Y_1 = 0 \qquad (5.41)$$

thus ensuring that the toroidal magnetic field will be small in the immediate vicinity of the boundary of the conducting medium.

It would also be possible in this approach to *assign* any desired value to the dimensionless decay constant γ. However, as we are interested in studying the temporal behaviour of the magnetic field in terms of the other parameters of the problem, we may choose instead to assign a particular value to one of the coefficients Y_n $(n > 1)$. In order to allow γ to be complex, we shall assume that

$$Y_5 = C = \text{constant} \qquad (5.42)$$

5.3.6 The oscillating dipole field - "boundary layer control" of frequency

For the case of a dipole external field, $n = 1$. Substituting this value into equation (5.39), and combining (5.39), (5.41), and (5.42) we obtain

$$\gamma = -\frac{4}{3}\left\{1 + \frac{7}{16}x + \frac{1}{4}\beta x\right\} + \pm\left\{\frac{16}{9}\left[1 + \frac{7}{16}x + \frac{1}{4}\beta x\right]^2 + 80\frac{C}{R_1} - \frac{20}{3}\left[1 + \frac{1}{2}x + \frac{3}{20}\beta x + \frac{1}{40}\xi\beta x\right]\right\}^{1/2} \tag{5.43}$$

where

$$x \equiv 2R_2/R_1 = R''(0)/R'(0) = -r_0\,\alpha''(r_0)/\alpha'(r_0) \tag{5.44}$$

$$\beta \equiv 3R_3/R_2 = R'''(0)/R''(0) = -r_0\,\alpha'''(r_0)/\alpha''(r_0) \tag{5.45}$$

$$\xi \equiv 4R_4/R_3 = R^{IV}(0)/R'''(0) = -r_0\,\alpha^{IV}(r_0)/\alpha'''(r_0) \tag{5.46}$$

If the coefficients R_1, R_2, and R_3 are chosen in such a way that

$$1 + \frac{7}{16}x + \frac{1}{4}\beta x = 0 \tag{5.47}$$

then

$$x = \frac{-4}{\beta + 7/4} \tag{5.48}$$

$$\beta x = -\left\{4 + \frac{7}{4}x\right\} \tag{5.49}$$

and

$$-\gamma^2 = \frac{8}{3}\left\{\frac{\beta(1-\xi/4) - 5/8}{\beta + 7/4}\right\} - 80\,\frac{C}{R_1} \tag{5.49}$$

$$= \frac{8}{3}\left\{(1-\xi/4) + \frac{19}{32}(1 - 7\xi/38)\,x\right\} - 80\,\frac{C}{R_1} \tag{5.49'}$$

The imaginary part of γ provides a measure of the *oscillation frequency* of the external magnetic dipole field. If γ is purely imaginary, this frequency can be interpreted as the frequency of *polarity reversals*. From the definition of γ, (5.21), we see that the number of polarity reversals per million years is

$$\text{no. of reversals/m.y.} \approx \frac{1}{\pi}(\eta/r_o^2)\sqrt{-\gamma^2}\cdot\{3\times 10^{13}\ \text{sec/m.y.}\} \tag{5.50}$$

if γ is purely imaginary.

For the Earth, $\eta \simeq 3\ m^2/sec$ and $r_o \simeq 3 \times 10^6\ m$. Substituting these values into (5.50), we obtain

$$\text{no. of reversals/m.y.} \approx \frac{10}{\pi}\sqrt{-\gamma^2} \tag{5.51}$$

In *Figure 23* the number of reversals per million years given by (5.51) is plotted as a function of x [i.e. of $-r_o\alpha''(r_o)/\alpha'(r_o)$] for the case in which

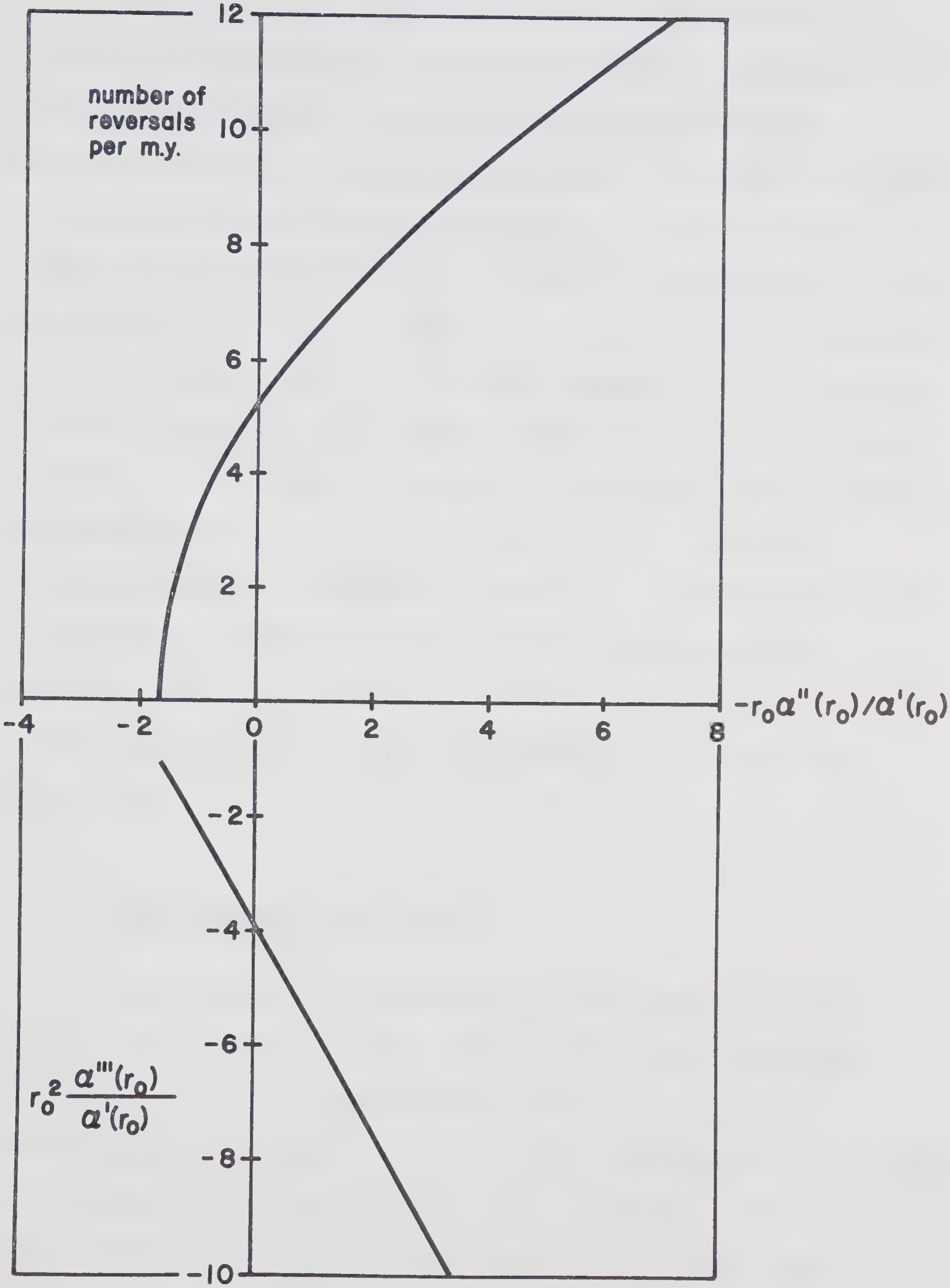
$$\xi = 0 = C \tag{5.52}$$

The value of βx [i.e. $-r_o^2\alpha'''(r_o)/\alpha'(r_o)$] given by (5.48') is also plotted as a function of x.

<u>Figure 23</u>. Parameters for the $\alpha^2(r)$ dynamo in a spherical shell as functions of $-r_o\alpha''(r_o)/\alpha'(r_o)$.

Upper curve: No. of reversals per million years, given by *equation (5.51)*, as a function of $x = -r_o\alpha''(r_o)/\alpha'(r_o)$. (In this plot, η and r_o have been assigned values appropriate to the Earth's fluid core.)

Lower curve: $\beta = r_o^2\alpha'''(r_o)/\alpha'(r_o)$ as a function of $x = -r_o\alpha''(r_o)/\alpha'(r_o)$.


number of reversals per m.y.
12
10
8
6
4
2
-4
-2
0
2
4
6
8
$-r_o\alpha''(r_o)/\alpha'(r_o)$
-2
-4
-6
-8
-10
$r_o^2\frac{\alpha'''(r_o)}{\alpha'(r_o)}$

The curves plotted in *Figure 23* correspond to a kinematic dynamo model in which the temporal behaviour of the external magnetic dipole field is controlled by turbulent motions near the boundary of the conducting fluid. A stationary state, corresponding to $\gamma = 0$ in (5.49), is changed to an oscillatory one by small fluctuations in the dependence of α on r near $r = r_o$. Of course, it must not be forgotten that α has been assumed *time-independent* in the solution of the induction equation described above. However, we may think in terms of a *sporadic* model in which the dependence of α on r near $r = r_o$ changes discontinuously at irregular intervals. It is interesting to note that values of the reversal frequency on the steepest part of the curve in *Figure 23* correspond closely to the values observed for the geomagnetic field (see *Figure 22*).

5.3.7 Consistency requirements

In order for the kinematic dynamo model described above to be consistent, the intensity of the turbulence in $r^* \leqslant r \leqslant r_o$ must be sufficiently great to allow the boundary conditions at $r = r^*$ to be satisfied by adjusting the values of the coefficients R_n. From (5.29)-(5.39) and (5.41)-(5.49), we see that when $\gamma = 0$ the values of X and Y at $z = z^*$ are given by

$$X(z^*) = 1 + z^* + (z^*)^2 + (z^*)^3 + (z^*)^4 + \{1 - \tfrac{1}{40}R_1^2\}(z^*)^5 + \dots \tag{5.53}$$

$$Y(z^*) = \tfrac{1}{2}R_1(z^*)^2\left\{1 + \tfrac{2}{19}z^* + \tfrac{1}{57}(z^*)^2 + \left[\tfrac{1}{3}\tfrac{R_5}{R_1} - \tfrac{1}{171}\right](z^*)^4 + \dots\right\} \tag{5.54}$$

If we assume that the boundary conditions on continuity of X and Y at $z = z^*$ have been satisfied by adjusting the parameters of the solution to the induction equation in the region $z^* \leqslant z \leqslant 1$, it follows from (5.53) and (5.54) that the conditions on continuity of X' and Y' at $z = z^*$ can only be satisfied by adjusting R_1 and R_5 if

$$\tfrac{1}{8}R_1^2(z^*)^4 \sim \mathcal{O}(1) \tag{5.55}$$

and

$$\frac{R_5}{R_1}(z^*)^4 \sim \mathcal{O}(1) \tag{5.56}$$

Steenbeck, Krause, and Rädler (1966) have estimated the value of α when a gradient of turbulent intensity is present in a system rotating with angular velocity $\underset{\sim}{\Omega}$.

$$\alpha \approx -\frac{u'\lambda_c^2\tau_c}{\eta}\,\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}u' \tag{5.57}$$

We shall use this estimate as a rough guide, despite the objections raised by *Lerche (1972e)* to the turbulence spectrum tensor used by *Steenbeck, Krause and Rädler*. In (5.57) u' is the turbulent intensity, λ_c the correlation length of the turbulence, and τ_c the correlation time.

For the case of the Earth we may write, very approximately,

$$-\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u} \;\sim\; \frac{\Omega\, u'}{r_0\, z^*} \tag{5.58}$$

ignoring the departure from spherical symmetry introduced by the presence of a preferred axis. Also, from (5.26) and (5.22),

$$R(z^*) \;=\; \frac{\alpha(r^*)r_0}{\eta} \;\approx\; R_1 z^* \tag{5.59}$$

Combining (5.57)-(5.59) and applying the condition (5.55), we obtain

$$R_1\cdot\{z^*\}^2 \;\sim\; \left\{\frac{u'\lambda_c}{\eta}\right\}^2 \tau_c\Omega \;\sim\; \mathcal{O}(1) \tag{5.60}$$

or

$$u'\lambda_c\tau_c^{1/2} \;\sim\; \eta/\Omega^{1/2} \tag{5.61}$$

For the Earth, $\eta \simeq 3\ \mathrm{m}^2/\mathrm{sec}$ and $\Omega \simeq 7 \times 10^{-5}$ rad/sec . Substituting these values into (5.61), we have

$$u'\lambda_c\tau_c^{1/2} \;\sim\; 4\times 10^2\ \mathrm{m}^2/\mathrm{sec}^{1/2} \tag{5.62}$$

Taking $u' \sim 10^{-4}$ m/sec as a reasonable estimate of the velocity near the surface of the core *(Elsasser, 1950; Busse, 1971; Roberts and Soward, 1972)*, we find that the condition (5.62) reduces to

$$\lambda_c\tau_c^{1/2} \;\sim\; 4\times 10^6\ \mathrm{m}\cdot\mathrm{sec}^{1/2} \tag{5.63}$$

In order to determine whether or not (5.63) can be satisfied in the Earth's fluid core, we must estimate

the values of λ_c and τ_c appropriate to turbulence near the core-mantle interface. We must also decide whether or not 10^{-4} m/sec is a reasonable estimate for turbulent velocities in this region. In this connection, it may be noted that *Steenbeck and Krause (1966)* use the values

$$u' \sim 10^{-2}\text{-}10^{-1} \text{ m/sec}$$
$$\lambda_c \sim 10^2 \text{-} 10^3 \text{ m}$$
$$\tau_c \sim 3 \text{ hours} \sim 10^4 \text{ sec}$$

for a general distribution of turbulence in the Earth's core. These values give

$$10^2 \text{ m}^2\text{sec}^{-1/2} < u'\lambda_c\tau_c^{1/2} < 10^4 \text{ m}^2\text{sec}^{-1/2}$$

a range which includes the value 4×10^2 m^{2}sec$^{-1/2}$ required by (5.62). On the other hand, if 10^{-4} m/sec is accepted as a reasonable value for u' near the core-mantle interface, (5.63) implies that τ_c and λ_c are related in the manner outlined in *Table 16*.

Turbulence near the outer boundary of the Earth's fluid core might well be associated with "bumps" on the core-mantle interface (Hide, 1967; *Hide and Horai, 1968; Hide, 1969a; Hide and Malin, 1970, 1971a,b,c)*, estimated to have horizontal dimensions $\sim 10^5$ m and vertical dimensions $\sim 10^3$ m *(Hide, 1969a; see Acheson and Hide, 1973)*. It may be seen from *Table 16* that values of λ_c in the range 10^3-10^2 m, corresponding to the estimated vertical dimensions of the

core-mantle "bumps", imply values of τ_c in the range 1-100 years. This is the range of time scales suggested by *Braginskii (1964d, 1970a,b, 1971)* and *Currie (1968)* for turbulent processes near the core-mantle boundary.

TABLE 16

Relationship between τ_c and λ_c implied by (5.63)

τ_c	(seconds)	λ_c (metres)
1 hour	3×10^3	7×10^4
1 day	9×10^4	1×10^4
1 month	3×10^6	2×10^3
1 year	3×10^7	7×10^2
10 years	3×10^8	2×10^2
100 years	3×10^9	7×10^1
1000 years	3×10^{10}	2×10^1

5.3.8 Objections to the $\alpha^2(r)$ model for geomagnetic reversals

The kinematic $\alpha^2(r)$ dynamo model in a spherical shell, described in the last few sections, is not intended as a serious explanation of how polarity reversals occur in the geodynamo. Several important objections to the model can be raised.

a. The character of reversals in the model is considerably different from the irregular nature of geomagnetic reversals. The "geomagnetic reversal frequency" plotted in *Figure 22* is really an *average* quantity. The durations of geomagnetic polarity intervals fluctuate widely about the mean. Furthermore, the *transition time* for a geomagnetic reversal is generally much shorter than the time between reversals *(see section 5.1.2)*.

b. The effects of *rotation* have been ignored. It is expected that in the Earth's core α will depend on θ as well as on r. As indicated in (5.57), α will be proportional to $\cos\theta$ if the gradient of turbulent intensity is radial and the axis of rotation is taken as the z-axis.

c. It is perhaps unrealistic to assume that the external dipole field of the geodynamo is due principally to turbulent motions near the core-mantle interface.

d. No attempt has been made to include *hydromagnetic effects* in the model.

It must be emphasized that objection (d) applies to *any* kinematic model for geomagnetic reversals. If no stringent restrictions are placed on the form of the velocity field, reversals can be made to occur in many different ways - *Levy's* model provides one example, the present model another. Unless the *hydromagnetic dynamo equations* are considered, no firm conclusions can be drawn about the validity of any given model.

We may note, however, that the idea of *boundary-layer control* of the temporal behaviour of the geomagnetic field is worthy of closer investigation. This subject will be considered in greater detail in the remainder of this chapter.

5.4 Integral properties of the kinematic dynamo equations

5.4.1 Introduction

One of the most useful approaches to the study of reversals and other temporal variations of the geomagnetic field has been the *homopolar disk dynamo* analogy proposed by *Bullard (1955)*. In this model, the problems of boundary conditions and fluid motions are set aside on the assumption that the feature of the *hydromagnetic dynamo* most relevant to the temporal behaviour of the magnetic field is the *nonlinearity* of the interaction between the driving forces and the magnetic field.

Although a single disk dynamo of the type studied by *Bullard (1955)* does not exhibit reversals *(but see Malkus, 1972c)*, the magnetic field of a system of two or more *coupled disk dynamos* reverses and oscillates in an irregular manner remarkably similar to the observed behaviour of the geomagnetic dipole field *(Rikitake, 1958; Lowes, 1960; Lebovitz, 1960; Allan, 1962; Mathews and Gardner, 1963; Somerville, 1967; Suffolk, 1970; Cook and Roberts, 1970; Cook, 1972; Bullard and Gubbins, 1971, 1973; Malkus, 1972c).* The principal advantage of studying a system of this type is its mathematical simplicity. The dynamo equations are replaced by a finite system of coupled ordinary differential equations, with time as the independent variable. It would clearly be very useful if the hydromagnetic

dynamo equations could be reduced to a system of this type.

Lorenz (1960, 1962) has proposed methods of reducing a system of coupled partial differential equations to a set of equations of the *disk dynamo* type, but his approach is not readily applicable to the hydromagnetic dynamo problem. It would appear to be more useful to examine the *integral properties* of the dynamo equations, particularly as the quantities of interest, such as the *magnetic dipole moment* of a current distribution, are themselves integral quantities. Some progress in this direction has been made by *Runcorn (1955)* and *Backus (1958)*.

In the remainder of this chapter, we shall consider the integral properties of the kinematic dynamo equations. Consideration of hydromagnetic effects will be deferred until *Chapter 6*.

5.4.2 Multipole representation of the external fields

The electric and magnetic fields in a non-conducting medium surrounding a conductor can be represented in terms of the *electric and magnetic multipole moments* of the charge and current distributions in the conducting medium. If the gauge

$$\operatorname{div} \underset{\sim}{A} = 0 \tag{5.64}$$

is used in the *quasi-steady approximation*, where $\underset{\sim}{A}$ is the

vector potential defined in (1.58), these multipole expansions can be written in the form

$$\hat{\underset{\sim}{E}} = -\frac{1}{4\pi\epsilon}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\,Q^{(m)}\cdot\nabla^m\left(\underset{\sim}{\nabla}\frac{1}{r}\right) + \tag{5.65}$$

$$-\frac{\mu}{4\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\,\dot{T}^{(m)}\cdot\nabla^{m-1}\times\underset{\sim}{\nabla}\frac{1}{r}$$

$$\hat{\underset{\sim}{B}} = -\frac{\mu}{4\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\,T^{(m)}\cdot\nabla^m\left(\underset{\sim}{\nabla}\frac{1}{r}\right) \tag{5.66}$$

where

$$Q^{(m)} \equiv \int_V \theta\,\underset{\sim}{r}^m\,dv \tag{5.67}$$

and

$$T^{(m)} \equiv \frac{m}{m+1}\int_V (\underset{\sim}{r}\times\underset{\sim}{j})\,\underset{\sim}{r}^{m-1}\,dv \tag{5.68}$$

are the *electric and magnetic 2^m-pole moments*, and $\dot{T}^{(m)}$ denotes $dT^{(m)}/dt$. In (5.67) and (5.68), the integration is taken over the volume of the charge and current distribution. In (5.65)-(5.68) we have used the notation

$$\mathbb{F}^{(m)}\cdot\nabla^m \equiv \mathbb{F}^{(m)}_{a_1a_2\cdots a_m}\frac{\partial}{\partial x_{a_1}}\frac{\partial}{\partial x_{a_2}}\cdots\frac{\partial}{\partial x_{a_m}} \tag{5.69}$$

$$\underset{\sim}{r}^m \equiv x_{a_1}x_{a_2}\cdots x_{a_m}\,\underset{\sim}{1}_{a_1}\underset{\sim}{1}_{a_2}\cdots\underset{\sim}{1}_{a_m} \tag{5.70}$$

where the x_{a_i} are Cartesian components of the position vector $\underset{\sim}{r}$, the $\underset{\sim}{1}_{a_i}$ are unit vectors, and the summation convention is implied in each case. *(See Appendix 2 for the*

derivation of these equations.) It should be noted that θ and $\underset{\sim}{j}$ in equations (5.67) and (5.68) are the *charge and current densities* defined in *section 1.3.1.*

5.4.3 Magnetic and electric multipole moments - representation in terms of the internal magnetic and velocity fields

From equations (1.4) and (1.6) we have

$$\theta = \epsilon\, \underset{\sim}{\nabla} \cdot \underset{\sim}{E} \tag{5.71}$$

so that, by (1.15)

$$\theta = \epsilon\, \underset{\sim}{\nabla} \cdot \left\{ \frac{1}{\sigma} \underset{\sim}{j} - \underset{\sim}{u} \times \underset{\sim}{B} \right\} \tag{5.72}$$

Since $$\underset{\sim}{\nabla} \cdot \underset{\sim}{j} = 0 \tag{5.73}$$

in the quasi-steady approximation, (5.72) reduces to

$$\theta = -\epsilon\, \underset{\sim}{\nabla} \cdot \{ \underset{\sim}{u} \times \underset{\sim}{B} \} \tag{5.74}$$

and (5.67) becomes

$$Q^{(m)} = -\epsilon \int_V \underset{\sim}{\nabla} \cdot (\underset{\sim}{u} \times \underset{\sim}{B})\, \underset{\sim}{r}^m\, dv \tag{5.75}$$

Equation (5.75) can be further simplified by making use of the *no-slip condition* $[\underset{\sim}{n} \times \underset{\sim}{u}] = 0$ on the boundary of V *(see section 1.7.2)* to obtain

$$Q^{(m)} = -\epsilon\int_V \{ \underset{\sim}{\nabla}\cdot[(\underset{\sim}{u}\times\underset{\sim}{B})\underset{\sim}{r}^m] - (\underset{\sim}{u}\times\underset{\sim}{B})\cdot\underset{\sim}{\nabla}(\underset{\sim}{r}^m)\} \, dv$$

$$= -\epsilon\int_S (\underset{\sim}{n}\times\underset{\sim}{u})\cdot\underset{\sim}{B}\,\underset{\sim}{r}^m \, dS + \epsilon\int_V (\underset{\sim}{u}\times\underset{\sim}{B})\cdot\underset{\sim}{\nabla}(\underset{\sim}{r}^m)\, dv$$

$$= \epsilon\int_V (\underset{\sim}{u}\times\underset{\sim}{B})\cdot\underset{\sim}{\nabla}(\underset{\sim}{r}^m)\, dv \tag{5.76}$$

Equation (5.76) expresses the electric multipole moment tensor $Q^{(m)}$ in terms of the fields $\underset{\sim}{u}$ and $\underset{\sim}{B}$ rather than their derivatives.

In *Appendix 2* a similar representation is derived for the magnetic multipole tensor, expressing $T^{(m)}$ in terms of $\underset{\sim}{B}$ rather than in terms of $\underset{\sim}{j}$. When the conducting volume V is spherical, the expressions for the first few magnetic multipole moments become

[dipole moment]

$$\underset{\sim}{T}^{(1)} = \frac{3}{2\mu}\int_V \underset{\sim}{B}\, dv \tag{5.77}$$

[quadrupole moment]

$$\underset{\approx}{T}^{(2)} = \frac{2}{3\mu}\int_V \{ 4\underset{\sim}{B}\underset{\sim}{r} + \underset{\sim}{r}\underset{\sim}{B} - (\underset{\sim}{r}\cdot\underset{\sim}{B})\underset{\approx}{I}\}\, dv \tag{5.78}$$

[octupole moment]

$$\underset{\approx}{T}^{(3)} = \frac{3}{4\mu}\int_V \{ 5\underset{\sim}{B}\underset{\sim}{r}\underset{\sim}{r} + \underset{\sim}{r}\underset{\sim}{B}\underset{\sim}{r} + \underset{\sim}{r}\underset{\sim}{r}\underset{\sim}{B} - (\underset{\sim}{r}\cdot\underset{\sim}{B})(\underset{\approx}{I}\underset{\sim}{r} + \underset{\sim}{1}_i\underset{\sim}{r}\underset{\sim}{1}_i)\}\, dv + \tag{5.79}$$

$$+ \frac{4}{5}\mu r_0^2 \{ \tfrac{2}{3}\underset{\sim}{T}^{(1)}\underset{\approx}{I} - \underset{\sim}{1}_i\underset{\sim}{T}^{(1)}\underset{\sim}{1}_i - \underset{\approx}{I}\underset{\sim}{T}^{(1)}\}$$

In general, the expression for the 2^{2n}-pole moment involves a summation over the 2^{2i}-pole moments, and the expression for the 2^{2n+1}-pole moment involves a summation over the 2^{2i+1}-pole moments, where $i = 1,2,\dots,n-1$ in each case. For example, the expression for the *sedecimupole moment tensor (Winch, 1967a)* $T^{(4)}$ will contain terms involving the quadrupole moment tensor $T^{(2)}$, and the expression for the *duotrigintupole moment tensor (Winch, 1967b)* $T^{(5)}$ will contain terms involving the dipole and octupole moment tensors $T^{(1)}$ and $T^{(3)}$.

5.4.4 Temporal behaviour of the external magnetic dipole moment

The problem of the temporal variation of the external magnetic field of a current distribution may now be studied with the aid of equations (5.76)-(5.79). Taking the time derivative of equation (5.77), and making use of the magnetic induction equation (1.16') and the boundary condition $\underset{\sim}{u} = 0$ on S *(see section 1.7.2)*, we obtain

$$\dot{\underset{\sim}{T}}^{(1)} = \frac{3}{2\mu}\int_V \{\partial\underset{\sim}{B}/\partial t\}\, dv$$

$$= \frac{3}{2\mu}\int_V \{\eta\nabla^2\underset{\sim}{B} + \underset{\sim}{\nabla}\times(\underset{\sim}{u}\times\underset{\sim}{B})\}\, dv$$

$$= \frac{3}{2\mu}\int_V \eta\nabla^2\underset{\sim}{B}\, dv + \frac{3}{2\mu}\int_S \underset{\sim}{n}\times(\underset{\sim}{u}\times\underset{\sim}{B})\, dS$$

or

$$\dot{\underset{\sim}{T}}^{(1)} = \frac{3\eta}{2\mu}\int_V \nabla^2 \underset{\sim}{B}\, dv \quad = -\frac{3\eta}{2}\int_S \underset{\sim}{n}\times\underset{\sim}{j}\, dS \tag{5.80}$$

From (5.80) we see that the rate of change of the magnetic dipole moment depends *only* on the dissipation term $\eta\nabla^2\underset{\sim}{B}$ in the induction equation, and thus has no direct link with either the velocity field $\underset{\sim}{u}$ or the external potential field $\hat{\underset{\sim}{B}}$. Differentiating again with respect to time, we obtain

$$\begin{aligned} \ddot{\underset{\sim}{T}}^{(1)} &= \frac{3\eta}{2\mu}\int_V \nabla^2\left\{\eta\nabla^2\underset{\sim}{B} + \underset{\sim}{\nabla}\times(\underset{\sim}{u}\times\underset{\sim}{B})\right\} dv \\ &= \frac{3\eta^2}{2\mu}\int_V \nabla^4\underset{\sim}{B} \; + \frac{3\eta}{2\mu}\int_S \underset{\sim}{n}\times\nabla^2(\underset{\sim}{u}\times\underset{\sim}{B})\, dS \end{aligned} \tag{5.81}$$

The second time derivative of the magnetic dipole moment is thus directly linked to both the dissipative and the regenerative terms in the induction equation.

5.4.5 "Boundary-layer control" of the external dipole moment

Expanding the regenerative term in (5.81), we obtain

$$\ddot{T}_i^{(1)} = \frac{3\eta^2}{2\mu}\int_V \nabla^4 B_i\, dv \; + \tag{5.82}$$

$$+\frac{3\eta}{2\mu}\epsilon_{ijm}\epsilon_{m\ell k}\int_S n_j\left\{B_k\nabla^2 u_\ell + 2\frac{\partial u_\ell}{\partial x_r}\frac{\partial B_k}{\partial x_r}\right\} dS$$

The three integrals on the right hand side of equation (5.82) can be estimated as follows:

$$I^{(1)} \equiv \left| \frac{3\eta^2}{2\mu} \int_V \nabla^4 \underset{\sim}{B}\, dv \right| \sim \frac{3\eta^2}{2\mu} \cdot \frac{4\pi}{3} r_o^3 \cdot \frac{B_o}{L^4} \quad (5.83)$$

$$I^{(2)} \equiv \left| \frac{3\eta}{2\mu} \int_S \underset{\sim}{n}\, \underset{\sim}{B}\, \nabla^2 \underset{\sim}{u}\, dS \right| \sim \frac{3\eta}{2\mu} \cdot 4\pi r_o^2 \cdot B_o^{(S)} \cdot \frac{u_o}{\delta^2} \quad (5.83')$$

$$I^{(3)} \equiv \left| \frac{3\eta}{2\mu} \int_S 2 \underset{\sim}{n} \frac{\partial \underset{\sim}{u}}{\partial x_r} \frac{\partial \underset{\sim}{B}}{\partial x_r} dS \right| \sim \frac{3\eta}{\mu} \cdot 4\pi r_o^2 \cdot \frac{B_o^{(S)}}{L} \cdot \frac{u_o}{\delta} \quad (5.83")$$

where B_o is the average magnitude of the magnetic flux density in V, $B_o^{(S)}$ the average magnitude of the flux density on S, and L a length scale defined in such a way as to make B_o/L^2 the average value of $|\nabla^2 B|$ in V. r_o is the radius of the spherical volume V, δ the thickness of the boundary layer on S (assuming that one exists), and u_o the average change in velocity across the boundary layer.

Comparing terms in (5.83)-(5.83"), we have

$$I^{(1)}/I^{(2)} \sim \frac{\eta r_o}{3u_o} \cdot \frac{B_o}{B_o^{(S)}} \cdot \frac{\delta^2}{L^4} \quad (5.84)$$

$$I^{(3)}/I^{(2)} \sim 2\, \delta/L \quad (5.84')$$

For the Earth, we may take

$$B_o/B_o^{(S)} \lesssim 10^2 \quad (5.85)$$

$$r_o \sim 3 \times 10^6 \text{ m} \quad (5.86)$$

$$\eta \sim 3 \text{ m}^2/\text{sec} \quad (5.87)$$

Substituting these values into (5.84), we obtain

$$I^{(1)}/I^{(2)} \lesssim \{3 \times 10^{8}\ \mathrm{m^3/sec}\}\ \delta^2/u_o L^4 \tag{5.88}$$

From equations (5.84') and (5.88) we see that if

$$\delta/L \ll 1 \tag{5.89}$$

and
$$\delta^2/u_o L^4 \ll \{3 \times 10^{-9}\ \mathrm{sec/m^3}\} \tag{5.90}$$

the dominant term in (5.81) will be the one involving $\nabla^2 \underset{\sim}{u}$. Under these conditions we may write, approximately,

$$\ddot{T}_i^{(1)} \approx \frac{3\eta}{2\mu} \epsilon_{ijm} \epsilon_{m\ell k} \int_S n_j B_k \nabla^2 u_\ell \, dS \tag{5.91}$$

When equation (5.91) is valid, the temporal behaviour of the geomagnetic dipole moment is *"controlled"* by the velocity distribution in the boundary layer.

In deriving equation (5.91) we have ignored the possibility of turbulence near the core-mantle boundary. If turbulence is present an appropriate "mean field" equation is obtained from equation (5.81) by writing $\overline{\underset{\sim}{B}}$ in place of $\underset{\sim}{B}$ and $\{\overline{\underset{\sim}{u}} \times \overline{\underset{\sim}{B}} + \overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}\}$ in place of $\underset{\sim}{u} \times \underset{\sim}{B}$ on the right hand side. Assuming for simplicity that $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'} = \alpha\overline{\underset{\sim}{B}}$, giving an α-*effect*, we find that equation (5.91) must be replaced by

$$\ddot{T}_i^{(1)} \approx \frac{3\eta}{2\mu} \int_S \Big\{ (\underset{\sim}{n} \cdot \overline{\underset{\sim}{B}}) \nabla^2 \overline{u}_i - (\underset{\sim}{n} \cdot \nabla^2 \overline{\underset{\sim}{u}}) \overline{B}_i + \epsilon_{ijk} (\nabla^2 \alpha) n_j \overline{B}_k \Big\} \, dS \tag{5.92}$$

if we make the additional assumption that the term involving $\nabla^2\alpha$ makes a contribution of the same order of magnitude as that involving $\nabla^2\bar{\underset{\sim}{u}}$.

5.4.6 Simplification of notation

For simplicity in writing complicated expressions, we shall make use of the notation

$$\partial_{a_i} \equiv \partial/\partial x_{a_i} \tag{5.93}$$

$$T^{(m)} \cdot \nabla^m \equiv T^{(m)}_{a_1 \cdots a_m} \partial_{a_1} \cdots \partial_{a_m} \tag{5.94}$$

$$\underset{\sim}{n} \equiv n_k \underset{\sim}{1}_k = \underset{\sim}{1}_r \tag{5.95}$$

We shall also make use of several theorems proved in *Appendix 2*. In particular,

$$\underset{\sim}{n} \cdot \underset{\sim}{\nabla} \left\{ \partial_{a_1} \cdots \partial_{a_m} \frac{1}{r} \right\} = -\frac{(m+1)}{r_0} \left\{ \partial_{a_1} \cdots \partial_{a_m} \frac{1}{r} \right\} \tag{5.96}$$

See *section A.2.3* for further details.

5.4.7 The "boundary-layer control" approximation

As it stands, equation (5.92) is not particularly useful, since it involves both $\underset{\sim}{T}^{(1)}$ and $\bar{\underset{\sim}{B}}$. However, since $\bar{\underset{\sim}{B}}$ is continuous across the boundary S *(see section 1.7.1)*, we may replace $\bar{\underset{\sim}{B}}$ in (5.92) with the multipole

expansion (5.66). Making this substitution and applying equation (5.96), we obtain

$$\ddot{T}_i^{(1)} \approx -\frac{3\eta}{8\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}T^{(m)}_{a_1\cdots a_m}\cdot \tag{5.97}$$

$$\cdot\int_S\Big\{-\frac{(m+1)}{r_o}(\nabla^2\bar{u}_i)\,\partial_{a_1}\cdots\partial_{a_m}\frac{1}{r} - (\underset{\sim}{n}\cdot\nabla^2\underset{\sim}{\bar{u}})\,\partial_{a_1}\cdots\partial_{a_m}\partial_i\frac{1}{r}$$

$$+\,\epsilon_{ijk}(\nabla^2\alpha)\,n_j\,\partial_{a_1}\cdots\partial_{a_m}\partial_k\frac{1}{r}\Big\}\,dS$$

Writing out the first few terms of (5.97) in detail, we have

$$\ddot{T}_i^{(1)} \approx \frac{3\eta}{8\pi}\Big\{T^{(1)}_{a_1}\int_S\Big[\frac{2}{r_o^3}(\nabla^2\bar{u}_i)\,n_{a_1} + \frac{1}{r_o^3}(\underset{\sim}{n}\cdot\nabla^2\underset{\sim}{\bar{u}})(\delta_{ia_1}-3n_in_{a_1})$$
$$-\frac{1}{r_o^3}\epsilon_{ija_1}n_j(\nabla^2\alpha)\Big]dS \;+$$
$$+\frac{1}{2}T^{(2)}_{a_1a_2}\int_S\Big[\frac{3}{r_o^4}(\nabla^2\bar{u}_i)(3n_{a_1}n_{a_2}-\delta_{a_1a_2}) \;+$$
$$+\frac{3}{r_o^4}(\underset{\sim}{n}\cdot\nabla^2\underset{\sim}{\bar{u}})(n_i\delta_{a_1a_2}+n_{a_1}\delta_{ia_2}+n_{a_2}\delta_{ia_1} - 5n_in_{a_1}n_{a_2})$$
$$-\frac{3}{r_o^4}(\nabla^2\alpha)\,\epsilon_{ijk}n_j(n_k\delta_{a_1a_2}+n_{a_1}\delta_{ka_2}+n_{a_2}\delta_{ka_1} - 5n_kn_{a_1}n_{a_2})\Big]dS$$
$$+\;\cdots\Big\} \tag{5.98}$$

If $|\underset{\approx}{T}^{(2)}/r_o| \ll |\underset{\sim}{T}^{(1)}|$, equation (5.98) reduces to

$$\ddot{T}_i^{(1)} \approx \frac{3\eta}{8\pi r_o^3}\,T^{(1)}_{a_1}\int_S\Big\{2n_{a_1}\nabla^2\bar{u}_i + \underset{\sim}{n}\cdot\nabla^2\underset{\sim}{\bar{u}}(\delta_{ia_1}-3n_in_{a_1}) - \epsilon_{ija_1}n_j\,\nabla^2\alpha\Big\}\,dS + \ldots \tag{5.99}$$

A further reduction is obtained if the usual *boundary layer approximation* for fluid motions in a sphere is made. In this approximation, tangential derivatives of $\underset{\sim}{u}$ are neglected in comparison with radial derivatives, and only the highest-order radial derivatives are retained, to first order. In addition, the component of velocity normal to the boundary is neglected in comparison with the tangential components. Thus,

$$\nabla^2 \overline{\underset{\sim}{u}} \approx \underset{\sim}{1}_\theta \frac{\partial^2}{\partial r^2} u_\theta + \underset{\sim}{1}_\phi \frac{\partial^2}{\partial r^2} u_\phi \tag{5.100}$$

$$\nabla^2 \alpha \approx \frac{\partial^2}{\partial r^2} \alpha \tag{5.101}$$

and equation (5.99) becomes

$$\ddot{\underset{\sim}{T}}^{(1)} \approx \frac{3\eta}{8\pi r_o^3} \int_S \left\{ 2\left(\underset{\sim}{1}_\theta \frac{\partial^2 u_\theta}{\partial r^2} + \underset{\sim}{1}_\phi \frac{\partial^2 u_\phi}{\partial r^2} \right)(\underset{\sim}{n} \cdot \underset{\sim}{T}^{(1)}) + - \frac{\partial^2 \alpha}{\partial r^2} \, \underset{\sim}{n} \times \underset{\sim}{T}^{(1)} \right\} dS + \cdots \tag{5.102}$$

Equation (5.102) will be referred to as the *boundary-layer control approximation*. The validity of this approximation depends on a number of assumptions, which will be summarized here for convenience.

a) $\delta/L \ll 1$ (5.89)

b) $\delta^2/u_o L^4 \ll \{3 \times 10^{-9} \text{ sec/m}^3\}$ (5.90)

c) $|\underset{\approx}{T}^{(2)}/r_o| \ll |\underset{\sim}{T}^{(1)}|$

The notation used in (5.89) and (5.90) is defined above

in connection with (5.83). (5.90) is the form of the assumption $I^{(1)} \ll I^{(2)}$ appropriate to the Earth's fluid core, where $I^{(1)}$ and $I^{(2)}$ are the integrals defined in (5.83) and (5.83'). It may be noted that (5.89) is more or less equivalent to the assumptions leading to the boundary-layer approximation, (5.100) and (5.101).

Within the framework of the kinematic dynamo problem, the *boundary-layer control approximation*, (5.102), gives a second-order ordinary differential equation for $T^{(1)}$, with time as the independent variable. When the hydromagnetic dynamo problem is considered, the coefficients of this equation depend on $T^{(1)}$, and the complexity of the equation is increased. This problem will be considered in *section 6.3*.

5.5 The boundary-layer control approximation and the geodynamo

5.5.1 Time-scale restrictions on boundary-layer control in the geodynamo

A rough estimate of the characteristic time scale of dipole moment variations governed by the *boundary-layer control approximation* may be obtained by scaling the terms in equation (5.102). When this is done, we find that the time scale τ is given by

$$\tau \sim \sqrt{r_0 \delta^2 / \eta u_0} \tag{5.103}$$

For the case of the geodynamo, $r_o \sim 3 \times 10^6$ m , $\eta \sim 3$ m^2/sec , and (5.103) becomes

$$\tau \sim 10^3 \cdot \sqrt{\delta^2 / u_0} \left\{ \frac{m \cdot m^2}{(m^2/s)(m/s)} \right\}^{1/2} \tag{5.103'}$$

From (5.103') we see that in order to get dipole moment variations in the geodynamo with a time scale of 10^K years, we must have

$$\delta^2 / u_0 \sim 10^{2K+9} \ m \cdot sec \tag{5.104}$$

Combining (5.104) with the assumption (5.90), we obtain a *consistency condition* for the boundary-layer control approximation in the geodynamo:

$$L > 3 \times 10^{4+K/2} \ m \tag{5.105}$$

In the geodynamo, L, defined as the length scale needed to make B_o/L^2 the "average value" *(defined in an integral sense - see section 5.4.5)* of $|\nabla^2 B|$ in the fluid core, is unlikely to be much larger than r_o, the core radius. Taking $r_o \sim 3 \times 10^6$ m as an upper bound on L in (5.105), we see that the limitation on κ is

$$\kappa < 4 \qquad (5.106)$$

The *boundary-layer control approximation* can therefore only be used to explain variations of the Earth's dipole moment which have a time scale shorter than 10^4 years.

5.5.2 Temporal variations of the Earth's magnetic dipole moment

As may be seen from *Table 13*, the geomagnetic dipole moment varies on several time scales in the range $\tau < 10^4$y. For example,

a) the dipole axis precesses, or "wobbles", around the axis of rotation in an irregular fashion, with a time scale of about 10^3 years *(see footnote* [6], *Table 13)*;

b) energy is transferred from the dipole field to higher-order multipole fields with a time scale of about 2×10^3 years *(see footnote* [4], *Table 13)*;

c) the rate at which energy is transferred from the dipole field to higher-order multipole fields varies

on a time scale of about 10^2 years *(see footnote* [5], *Table 13)*;

d) the direction and intensity of the dipole moment change on a time scale of 10^3-10^4 years during a polarity reversal *(see footnotes* [8], [9], *and* [10], *Table 13)*.

The "fundamental frequency" of dipole field strength oscillations corresponds to a period $\sim 10^4$ years. Since this period is so close to the maximum value of τ allowed by (5.106), it seems unlikely that this variation can be explained in terms of the boundary-layer control approximation.

Of the variations listed above, only (a) and (d) can be discussed in terms of the boundary-layer control approximation, equation (5.102). In order to discuss variations (b) and (c), we would have to return to equation (5.97), and retain higher-order multipole terms. In this thesis, we shall restrict consideration to equation (5.102).

5.5.3 Detailed expansion of the approximate dipole moment equation

Equation (5.102) can be written in the form

$$\begin{bmatrix} \ddot{T}_x^{(1)} \\ \ddot{T}_y^{(1)} \\ \ddot{T}_z^{(1)} \end{bmatrix} = \frac{3\eta}{4\pi r_0^3} \begin{bmatrix} (i_1 - i_3) & (i_4 - i_5 + i_6) & (i_{11} - i_{12} - i_8) \\ (i_4 + i_5 - i_6) & (i_2 + i_3) & (i_{13} + i_{14} + i_{10}) \\ (-i_7 + i_8) & (-i_9 - i_{10}) & (i_1 + i_2) \end{bmatrix} \cdot \begin{bmatrix} T_x^{(1)} \\ T_y^{(1)} \\ T_z^{(1)} \end{bmatrix} \tag{5.107}$$

where

$$i_1 \equiv \int_S \frac{\partial^2 \bar{u}_\theta}{\partial r^2} \cos\theta \sin\theta \cos^2\phi \; dS \tag{5.108}$$

$$i_2 \equiv \int_S \frac{\partial^2 \bar{u}_\theta}{\partial r^2} \cos\theta \sin\theta \sin^2\phi \; dS \tag{5.109}$$

$$i_3 \equiv \int_S \frac{\partial^2 \bar{u}_\phi}{\partial r^2} \sin\theta \sin\phi \cos\phi \; dS \tag{5.110}$$

$$i_4 \equiv \int_S \frac{\partial^2 \bar{u}_\theta}{\partial r^2} \cos\theta \sin\theta \cos\phi \sin\phi \; dS \tag{5.111}$$

$$i_5 \equiv \int_S \frac{\partial^2 \bar{u}_\phi}{\partial r^2} \sin\theta \sin^2\phi \; dS \tag{5.112}$$

$$i_6 \equiv \int_S \frac{1}{2} \frac{\partial^2 \alpha}{\partial r^2} \cos\theta \; dS \tag{5.113}$$

$$i_7 \equiv \int_S \frac{\partial^2 \bar{u}_\theta}{\partial r^2} \sin^2\theta \cos\phi \; dS \tag{5.114}$$

$$i_8 \equiv \int_S \frac{1}{2} \frac{\partial^2 \alpha}{\partial r^2} \sin\theta \sin\phi \; dS \tag{5.115}$$

$$i_9 \equiv \int_S \frac{\partial^2 \bar{u}_\theta}{\partial r^2} \sin^2\theta \sin\phi \; dS \tag{5.116}$$

$$i_{10} \equiv \int_S \frac{1}{2} \frac{\partial^2 \alpha}{\partial r^2} \sin\theta \cos\phi \, dS \tag{5.117}$$

$$i_{11} \equiv \int_S \frac{\partial^2 \overline{u}_\theta}{\partial r^2} \cos^2\theta \cos\phi \, dS \tag{5.118}$$

$$i_{12} \equiv \int_S \frac{\partial^2 \overline{u}_\phi}{\partial r^2} \cos\theta \sin\phi \, dS \tag{5.119}$$

$$i_{13} \equiv \int_S \frac{\partial^2 \overline{u}_\theta}{\partial r^2} \cos^2\theta \sin\phi \, dS \tag{5.120}$$

$$i_{14} \equiv \int_S \frac{\partial^2 \overline{u}_\phi}{\partial r^2} \cos\theta \cos\phi \, dS \tag{5.121}$$

Equation (5.107) may now be studied under various assumptions about the symmetry of the velocity field.

5.5.4 "Dipole wobble"

If $\overline{u}_\theta = 0$ and $\overline{u}_\phi$ and α are *axially symmetric* (i.e. independent of ϕ), equation (5.107) reduces to

$$\ddot{T}_x^{(1)} = -\left\{\frac{3\eta}{4\pi r_o^3}\right\}(i_5 - i_6)\, T_y^{(1)} \tag{5.122}$$

$$\ddot{T}_y^{(1)} = \left\{\frac{3\eta}{4\pi r_o^3}\right\}(i_5 - i_6)\, T_x^{(1)} \tag{5.122'}$$

Combining these two equations, and assuming that the time derivatives of $\overline{u}_\phi$ and α are small enough to be neglected, we obtain

$$\ddddot{T}_x^{(1)} = -\left\{\frac{3\eta}{4\pi r_o^3}\right\}^2 (i_5 - i_6)^2\, T_x^{(1)} \tag{5.123}$$

Solutions to equations (5.122) and (5.123) are of the form

$$T_x^{(1)} = a\, e^{\pm Kt} \cos Kt \qquad (5.124)$$

$$T_y^{(1)} = a\, e^{\pm Kt} \sin Kt \qquad (5.125)$$

where

$$K \equiv \sqrt{\frac{3\eta}{8\pi r_o^3}(i_5 - i_6)} \qquad (5.126)$$

These solutions indicate that the dipole moment vector precesses, or "wobbles", about the axis of symmetry, which may be taken as the axis of rotation in the case of the geodynamo. In (5.126)

$$i_5 \equiv \pi r_o^2 \int_0^{\pi} \frac{\partial^2 \overline{u}_\phi}{\partial r^2} \sin^2\theta \, d\theta \qquad (5.127)$$

$$i_6 \equiv \pi r_o^2 \int_0^{\pi} \frac{\partial^2 \alpha}{\partial r^2} \sin\theta \cos\theta \, d\theta \qquad (5.128)$$

Nonzero contributions to (5.127) and (5.128) arise from velocity components which satisfy

$$\overline{u}_\phi \propto P_{2n}(\cos\theta) \qquad (5.129a)$$

$$\alpha \propto P_{2n+1}(\cos\theta) \qquad (5.129b)$$

where the P_n are Legendre polynomials.

5.5.5 Axial dipole moment variations

If $\bar{u}_\phi = 0$ and $\bar{u}_\theta$ and α are axially symmetric (i.e. independent of ϕ), equation (5.107) reduces to

$$\ddot{T}_x^{(1)} = \left\{\frac{3\eta}{4\pi r_o^3}\right\}\left\{ i_1 T_x^{(1)} + i_6 T_y^{(1)} \right\} \tag{5.130}$$

$$\ddot{T}_y^{(1)} = \left\{\frac{3\eta}{4\pi r_o^3}\right\}\left\{ -i_6 T_x^{(1)} + i_1 T_y^{(1)} \right\} \tag{5.130'}$$

$$\ddot{T}_z^{(1)} = \left\{\frac{3\eta}{2\pi r_o^3}\right\}\left\{ i_1 T_z^{(1)} \right\} \tag{5.130"}$$

where

$$i_1 = \pi r_o^2 \int_S \frac{\partial^2 u_\theta}{\partial r^2} \cos\theta \sin^2\theta \, d\theta \tag{5.131}$$

$$i_6 = \pi r_o^2 \int_S \frac{\partial^2 \alpha}{\partial r^2} \cos\theta \sin\theta \, d\theta \tag{5.132}$$

Combining (5.130) and (5.130'), and neglecting time derivatives of $\bar{u}_\theta$ and α, we obtain

$$\ddddot{T}_x^{(1)} - \left\{\frac{3\eta}{2\pi r_o^3}\right\} i_1 \ddot{T}_x^{(1)} + \left\{\frac{3\eta}{4\pi r_o^3}\right\}^2 (i_1^2 + i_6^2) T_x^{(1)} = 0 \tag{5.133}$$

Equations (5.130), (5.130"), and (5.133) may now be solved for $T_x^{(1)}$, $T_y^{(1)}$, and $T_z^{(1)}$.

$$T_x^{(1)} = \mathcal{Re}\left\{ a_1 e^{K_1 t} + a_2 e^{-K_1 t} \right\} \tag{5.134}$$

$$T_y^{(1)} = \mathcal{Im}\left\{ a_1 e^{K_1 t} + a_2 e^{-K_1 t} \right\} \tag{5.135}$$

$$T_z^{(1)} = b_1 e^{K_2 t} + b_2 e^{-K_2 t} \tag{5.136}$$

where the a_i are complex constants, the b_i are real

constants, and

$$K_1 \equiv \sqrt{\frac{3\eta}{4\pi r_0^3}(i_1 - i\,i_6)} \tag{5.137}$$

$$K_2 \equiv \sqrt{\frac{3\eta}{2\pi r_0^3}\,i_1} \tag{5.138}$$

The solution given in (5.136) for $T_z^{(1)}$ requires $i_1 > 0$, so that K_2 is real. If $i_1 < 0$, (5.136) must be replaced with

$$T_z^{(1)} = \mathcal{R}e\left\{ b\, e^{K_2 t} \right\} \tag{5.136'}$$

where b is complex.

From equations (5.134)-(5.136') we see that an axisymmetric meridional flow near the boundary of the Earth's fluid core can produce large changes in the geomagnetic dipole moment on the time scale

$$\tau \sim \sqrt{r_0 \delta^2/\eta u_0} \tag{5.139}$$

As in *section 5.5.4*, the presence of an axisymmetric α-effect can lead to "dipole wobble", but an α-effect of this type has no effect on the axial dipole moment. As may be seen from (5.136'), the axial dipole moment $T_z^{(1)}$ can be made to "reverse", or even oscillate, if i_1 is negative. In contrast to the oscillatory reversals discussed in *section 5.3*, reversals in the *boundary-layer control* model are governed by the properties of the *mean*

flow, $\underset{\sim}{\bar{u}}$, rather than by the properties of the turbulence which gives rise to the α-effect.

As pointed out in (5.129b), i_6 is nonzero only if α satisfies

$$\alpha \propto P_{2n+1}(\cos\theta) \tag{5.140a}$$

Similarly, i_1 is nonzero only if

$$u_\theta \propto P_{2n}(\cos\theta) \tag{5.140b}$$

where the P_n are Legendre polynomials.

5.5.6 Limitations on the boundary-layer control approximation in the geodynamo

The principal limitation on the boundary-layer control approximation in the geodynamo is the *time-scale restriction* discussed in *section 5.5.1*. However, other limitations must also be considered.

The dipole moment solutions obtained in *sections 5.5.4 and 5.5.5* depend on the assumption that the velocity distribution is independent of time. If this is not the case, the behaviour of the dipole moment with time, within the framework of the boundary-layer control approximation, will be considerably more complicated. This aspect of the dipole moment variation is outside the scope of the kinematic dynamo problem. In order to determine the time

dependence of the velocity field, we must consider the *hydromagnetic dynamo problem. (See Chapter 6.)*

As noted in *section 5.5.2*, equation (5.102) does not permit discussion of the transfer of energy from the dipole field to higher-order multipoles. This problem can only be studied if the higher-order terms in (5.97) are retained, and expressions for the time derivatives of the higher-order multipole moments are obtained from equations (5.77)-(5.79). Unfortunately, simplifications of the type encountered in the derivation of equation (5.102) are not common. The quadrupole and octupole moment tensors $\underset{\approx}{T}^{(2)}$ and $\underset{\approx}{T}^{(3)}$ depend on the integral moments of the dissipative term in the induction equation, on the electric multipole moments, on the higher-order magnetic multipole moments, and so on. The integrals

$$\int_V (\underset{\sim}{u} \times \underset{\sim}{B})\, \underset{\sim}{\nabla}\, \underset{\sim}{r}^m \, dv$$

which must be evaluated if the higher-order moments are to be studied, do not have a useful representation in terms of the external potential fields. These integrals can only be treated sensibly within the context of the *hydromagnetic dynamo problem*.

The most serious disadvantage of the boundary-layer control approximation is the fact that *dissipative effects* are neglected. Although no attempt will be made here to

overcome this drawback, it may be possible to improve the approximation by including an estimate of the dissipative term in equation (5.82). Unfortunately, it is difficult to see how the time dependence of the term could be included in such an estimate.

Despite these limitations, we shall continue to use the boundary-layer control approximation in the next chapter. It is particularly encouraging that the *dipole wobble* appears to be well represented within the framework of the approximation. We shall consider this variation in more detail in *section 6.3.4*.

5.6 Summary of Chapter 5

This chapter is concerned with temporal variations of astrophysical magnetic fields, with particular reference to the geomagnetic field. A summary of observational knowledge of the temporal behaviour of astrophysical fields is presented in *section 5.1*.

The *$\alpha^2(r)$ kinematic dynamo in a spherical shell* is studied in detail in *section 5.3*. It is shown that regardless of the fluid motions in the deep interior of the sphere, it is possible to choose a turbulent velocity distribution near the outer boundary which makes the external magnetic dipole field vary with time in a periodic manner. The frequency of the dipole field oscillation is a sensitive function of the dependence of α on r at the boundary. Application of this model to the Earth is discussed, and several serious shortcomings are pointed out.

The idea of *boundary-layer control* of the external magnetic field of a spherical volume of conducting fluid is discussed for more general distributions of velocity in *section 5.4*. A set of expressions is presented which relate the multipole moments of a spherical distribution of currents to the integral moments of the internal magnetic field in a novel manner. These expressions are used to derive a differential equation for the external magnetic dipole moment as a function of time.

In *section 5.5* it is shown that *boundary-layer control in the geodynamo* is only possible for field variations with a time scale less than 10^4 years. It is also shown that *dipole wobble* and large variations of the axial dipole moment can be accounted for by certain distributions of velocity near the outer boundary of the Earth's fluid core.

6. MEAN FIELD ELECTRODYNAMICS AND THE HYDROMAGNETIC DYNAMO PROBLEM

6.1 The hydromagnetic dynamo problem

6.1.1 The dynamo equations

As was pointed out in *section 1.5*, the hydromagnetic dynamo problem is concerned with the simultaneous solution of the *electrodynamic* and the *hydrodynamic* equations - usually in a rotating system. These equations are summarized in *section 1.5.3*. Rewriting them here for convenience, we have

$$\left\{ \partial/\partial t - \eta \nabla^2 \right\} \underset{\sim}{B} = \operatorname{curl} \left\{ \underset{\sim}{u} \times \underset{\sim}{B} \right\} \tag{6.1}$$

$$\underset{\sim}{\nabla} \cdot \left\{ \rho \underset{\sim}{u} \right\} = 0 \tag{6.2}$$

$$\begin{aligned} \partial \underset{\sim}{u}/\partial t + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} \\ = -\frac{1}{\rho} \underset{\sim}{\nabla} P - \frac{1}{2\rho} |\underset{\sim}{\Omega} \times \underset{\sim}{r}|^2 \underset{\sim}{\nabla} \rho - \partial \underset{\sim}{\Omega}/\partial t \times \underset{\sim}{r} + \\ + \nu \left\{ \nabla^2 \underset{\sim}{u} + \underset{\sim}{\nabla} \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \right\} + \frac{\nu}{\rho} \left\{ \underset{\sim}{\nabla} \rho \cdot \underset{\sim}{\nabla} \underset{\sim}{u} + \underset{\sim}{\nabla} \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \rho \right\} + \\ + \frac{1}{\rho\mu} \underset{\sim}{B} \cdot \underset{\sim}{\nabla} \underset{\sim}{B} + \frac{1}{\rho} \underset{\sim}{F} \end{aligned} \tag{6.3}$$

$$\begin{aligned} = -\frac{1}{\rho} \underset{\sim}{\nabla} \left\{ P - \rho(\zeta - \tfrac{2}{3}\nu) \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \right\} - \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r}) - \partial \underset{\sim}{\Omega}/\partial t \times \underset{\sim}{r} + \\ + \nu \left\{ \nabla^2 \underset{\sim}{u} + \underset{\sim}{\nabla} \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \right\} + \frac{\nu}{\rho} \left\{ \underset{\sim}{\nabla} \rho \cdot \underset{\sim}{\nabla} \underset{\sim}{u} + \underset{\sim}{\nabla} \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \rho \right\} + \\ + \frac{1}{\rho\mu} \left\{ \underset{\sim}{\nabla} \times \underset{\sim}{B} \right\} \times \underset{\sim}{B} + \frac{1}{\rho} \underset{\sim}{F} \end{aligned} \tag{6.3'}$$

where

$$P \equiv p + \frac{B^2}{2\mu} - \rho(\zeta - \tfrac{2}{3}\nu)\underset{\sim}{\nabla}\cdot\underset{\sim}{u} - \tfrac{1}{2}\rho|\underset{\sim}{\Omega}\times\underset{\sim}{r}|^2 \tag{6.3"}$$

The magnetic flux density $\underset{\sim}{B}$ is subject to the condition

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{B} = 0 \tag{6.4}$$

At the boundary of the conducting fluid, which, in the case of the geodynamo is the core-mantle interface,

$$\underset{\sim}{u} = 0 \tag{6.5}$$

$$\langle\underset{\sim}{B}\rangle = 0 \tag{6.6}$$

$$\langle\underset{\sim}{n}\times\underset{\sim}{E}\rangle = 0 \tag{6.7}$$

In the conducting part of the solid mantle, the induction equation, (6.1), is replaced by the equation

$$\partial\underset{\sim}{B}/\partial t = \frac{1}{\mu\sigma_m}\nabla^2\underset{\sim}{B} \tag{6.8}$$

[where σ_m is the mantle conductivity], subject to the condition (6.4) and the boundary conditions (6.6) and (6.7). In the external nonconducting region, equation (6.8) reduces further to

$$\nabla^2\underset{\sim}{B} = 0 \tag{6.9}$$

subject to (6.4) and the condition

$$|\underset{\sim}{B}| \to \mathcal{O}(r^{-3}) \text{ as } r \to \infty \tag{6.10}$$

where r is measured from within the conducting region.

6.1.2 The neglect of inertial terms

In most studies of hydromagnetic planetary dynamos, the Navier-Stokes equation (6.3) is simplified by neglecting the first two terms on the left hand side *(Hide, 1956, 1966a; Taylor, 1963; Braginskii, 1964d, 1967b, 1970a,b, 1971; Malkus, 1963, 1967a,b; P.H. Roberts, 1967b, 1971a; Hide and Stewartson, 1972; Moffatt, 1970b, 1972; Acheson and Hide, 1973)*. Comparing these terms to the Coriolis force term, $2\underset{\sim}{\Omega} \times \underset{\sim}{u}$, we see that

$$\frac{|\partial \underset{\sim}{u}/\partial t|}{|2\underset{\sim}{\Omega} \times \underset{\sim}{u}|} \sim \frac{1}{2\Omega T} \tag{6.11}$$

$$\frac{|\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}|}{|2\underset{\sim}{\Omega} \times \underset{\sim}{u}|} \sim \frac{u}{2\Omega L} \tag{6.11'}$$

where L and T are the length and time scales of the variation of $\underset{\sim}{u}$.

For the case of the Earth, we may follow *Hide (1956)* and assume that

$$T \sim 10^2 \text{ years} \sim 3 \times 10^9 \text{ sec.} \tag{6.12}$$

$$L \sim 3 \times 10^6 \text{ m.} \tag{6.12'}$$

$$u \sim 10^{-3} \text{ m/sec.} \tag{6.12"}$$

$$\Omega \sim 7 \times 10^{-5} \text{ rad/sec.} \tag{6.12 }$$

[N.B. (6.12") may be an overestimate for u - *see, for example, Roberts and Soward, 1972.*] Substituting these

values into (6.11) and (6.11'), we have

$$\frac{1}{2\Omega T} \sim \frac{u}{2\Omega L} \sim 2 \times 10^{-6} \tag{6.13}$$

indicating that neglect of the inertial terms $\partial \underset{\sim}{u}/\partial t$ and $\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}$ in (6.3) and (6.3') is a reasonable approximation in the geodynamo.

6.1.3 The viscous boundary layer approximation

The ratio of the magnitudes of the viscous force term $\nu(\nabla^2\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{u})$ and the Coriolis force term $2\underset{\sim}{\Omega} \times \underset{\sim}{u}$ in (6.3) can be scaled as follows:

$$\frac{|\nu(\nabla^2\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{u})|}{|2\underset{\sim}{\Omega}\times\underset{\sim}{u}|} \sim \frac{\nu}{2\Omega L^2} \equiv \varepsilon \tag{6.14}$$

where ε is an *Ekman number*. For the case of the Earth, taking $\Omega \sim 7 \times 10^{-5}\ \mathrm{sec}^{-1}$ and $L \sim r_o \sim 3 \times 10^6$ m , we see that if $\nu \ll 10^9\ \mathrm{m}^2\mathrm{sec}^{-1}$, then $\varepsilon \ll 1$. Under these conditions, the flow in the main body of the core can be assumed *inviscid*.

As pointed out by *Hide (1971b)* and *Gans (1972a)*, the kinematic viscosity ν in the Earth's fluid core is one of the most obscure parameters of the Earth. Estimates in the literature range from $10^{-7}\ \mathrm{m}^2\mathrm{sec}^{-1}$ to $10^{+7}\ \mathrm{m}^2\mathrm{sec}^{-1}$ *(see Hide, 1971b; Gans, 1972b)*. However, in recent years arguments have been advanced which reduce the uncertainty in the

probable value of ν .

Gans (1972a) has calculated the value of ν at the boundary between the inner core and the outer core, assuming that this boundary is a melting transition, and using the *Andrade hypothesis*. His calculations indicate that in this region

$$2.8 \times 10^{-7}\ m^2 sec^{-1} < \nu < 1.5 \times 10^{-6}\ m^2 sec^{-1}$$

with a suggested typical value of $6 \times 10^{-7}\ m^2 sec^{-1}$. *Gans* points out that if the temperature gradient in the outer core is very shallow, as suggested by *Higgins and Kennedy (1971)*, the range of ν will be approximately the same throughout the outer core. If, on the other hand, a steeper temperature gradient is relevant, the value of ν in the body of the outer core will be *lower* than that at the inner core-outer core boundary. *Gans'* arguments are therefore summarized by the statement that

$$\nu \lesssim 6 \times 10^{-7}\ m^2 sec^{-1} \tag{6.15}$$

in the outer core.

Hide (1971b) has considered the value of ν at the core-mantle boundary. He argues that if "bumps" on the core-mantle interface strongly influence the flow in the core, as suggested by *Hide and Malin (1970, 1971a,b,c)*, their height must exceed the viscous boundary layer thickness by a certain factor. Using the estimate for the height

of these "bumps" provided by *Hide and Horai (1968)*, he finds that the *effective kinematic viscosity* (eddy plus molecular) at the core-mantle interface must satisfy

$$\nu_{eff} \lesssim 10^2 \ m^2 sec^{-1} \tag{6.15'}$$

It may be noted that seismic evidence indicates that compressional waves traverse the core without suffering appreciable attenuation *(Rochester, 1970)*. This result has led to the estimate that $\nu \sim 10^5 \ m^2 sec^{-1}$ in the outer core. *Gans (1972b)* has put forward the interesting speculation that a highly viscous region 5-10 km thick at the core-mantle interface, with $\nu \sim 10^7 \ m^2 sec^{-1}$, would allow the seismic result to be reproduced without upsetting the requirement that ν be very small in the body of the core.

If we accept *Gans'* estimate (6.15) for the kinematic viscosity in the body of the Earth's fluid core, we see from (6.14) that

$$\varepsilon \lesssim 5 \times 10^{-16} \tag{6.16}$$

Viscous forces will thus be negligible in the main body of the core, and the flow can be considered *inviscid*. However, since neglect of the viscous terms implies a lowering of the order of the Navier-Stokes equation (6.3), we can only satisfy the boundary condition (6.5) by assuming the existence of a *viscous boundary layer* at the surface of the core.

In the *viscous boundary layer approximation*, we may write

$$\underset{\sim}{u} = \underset{\sim}{u}^i + \underset{\sim}{u}^b \tag{6.17}$$

$$\underset{\sim}{B} = \underset{\sim}{B}^i + \underset{\sim}{B}^b \tag{6.18}$$

for the velocity and magnetic fields in the core. Here $(\underset{\sim}{u}^i, \underset{\sim}{B}^i)$ are the fields in the main body of the core, satisfying the *inviscid* equations, and $(\underset{\sim}{u}^b, \underset{\sim}{B}^b)$ are fields which adjust $\underset{\sim}{u}$ and $\underset{\sim}{B}$ to satisfy the boundary conditions at the core-mantle interface.

6.1.4 The approximate dynamo equations

If density gradients and temporal variations of the angular velocity Ω are neglected in (6.3), the equations for the fields $(\underset{\sim}{u}^i, \underset{\sim}{B}^i)$ become

$$\{\partial/\partial t - \eta\nabla^2\}\underset{\sim}{B}^i = \text{curl}\{\underset{\sim}{u}^i \times \underset{\sim}{B}^i\} \tag{6.19}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}^i = 0 \tag{6.20}$$

$$2\underset{\sim}{\Omega}\times\underset{\sim}{u}^i = -\frac{1}{\rho}\underset{\sim}{\nabla}P^i + \frac{1}{\rho\mu}\underset{\sim}{B}^i\cdot\underset{\sim}{\nabla}\underset{\sim}{B}^i + \frac{1}{\rho}\underset{\sim}{F} \tag{6.21}$$

$$= -\frac{1}{\rho}\underset{\sim}{\nabla}P_1 + \frac{1}{\rho\mu}(\underset{\sim}{\nabla}\times\underset{\sim}{B}^i)\times\underset{\sim}{B}^i + \frac{1}{\rho}\underset{\sim}{F} \tag{6.21'}$$

$$= \frac{1}{\rho}\{\underset{\sim}{F}_B^i + \underset{\sim}{F} - \underset{\sim}{\nabla}P_1\} \tag{6.21''}$$

where $\underset{\sim}{F}_B$ denotes the Lorentz force $\frac{1}{\mu}(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B}$, and

$$p_1 \equiv p - \tfrac{1}{2}\rho|\underset{\sim}{\Omega} \times \underset{\sim}{r}|^2 \qquad (6.22)$$

The *boundary layer fields* $(\underset{\sim}{u}^b, \underset{\sim}{B}^b)$ must satisfy the equations

$$\partial\underset{\sim}{B}^b/\partial t \approx \eta\,\partial^2\underset{\sim}{B}^b/\partial n^2 + B_n\,\partial\underset{\sim}{u}^b/\partial n - u_n\,\partial\underset{\sim}{B}^b/\partial n \qquad (6.23)$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}^b = 0 \qquad (6.24)$$

$$2\underset{\sim}{\Omega} \times \underset{\sim}{u}^b \approx \frac{1}{\rho\mu}\frac{\partial}{\partial n}\{\underset{\sim}{1}_n \times \underset{\sim}{B}^b\} \times \underset{\sim}{B} + \nu\,\partial^2\underset{\sim}{u}^b/\partial n^2 \qquad (6.25)$$

where we have made the usual boundary layer approximations,

$$\nabla^2\underset{\sim}{B}^b \approx \partial^2\underset{\sim}{B}^b/\partial n^2 \qquad (6.26)$$

$$\nabla^2\underset{\sim}{u}^b \approx \partial^2\underset{\sim}{u}^b/\partial n^2 \qquad (6.26')$$

$$\underset{\sim}{\nabla} \times \{\underset{\sim}{u} \times \underset{\sim}{B} - \underset{\sim}{u}^i \times \underset{\sim}{B}^i\} \approx B_n\,\partial\underset{\sim}{u}^b/\partial n - u_n\,\partial\underset{\sim}{B}^b/\partial n \qquad (6.26'')$$

In equations (6.23)-(6.26"), n is a coordinate normal to the boundary and directed into the fluid. The components B_n and u_n are given by

$$B_n = \underset{\sim}{B}\cdot\underset{\sim}{1}_n \qquad (6.27)$$

$$u_n = \underset{\sim}{u}\cdot\underset{\sim}{1}_n \qquad (6.27')$$

We have also made the assumption that $\underset{\sim}{u}^i$, $\underset{\sim}{B}^i$, $\underset{\sim}{F}$, and p_1 do not vary significantly across the boundary layer.

In equations (6.21) and (6.25) we have neglected the inertial terms, as discussed in *section 6.1.2*. The

viscous force terms, discussed in *section 6.1.3*, have been neglected in equation (6.21). However, these terms are retained in equation (6.25), as is necessary in a *boundary layer approximation*. Equations (6.19)-(6.21") are frequently referred to as the equations of the *magneto-geostrophic approximation (see section 1.9.1)*.

6.2 Solution of the approximate Navier-Stokes equation in a sphere

6.2.1 Preliminary manipulation

Taking the vector cross product of $\underset{\sim}{\Omega}$ with equation (6.21"), and introducing a cylindrical system of coordinates $[\varpi,\phi,z]$ with the z-axis directed along the axis of rotation, we obtain

$$\rho \underset{\sim}{u}^{i} = \frac{1}{2\Omega}\left\{(\underset{\sim}{F}_{B}^{i} + \underset{\sim}{F}) \times \underset{\sim}{1}_{z} - \underset{\sim}{\nabla} p_{1} \times \underset{\sim}{1}_{z}\right\} + \rho u_{z}^{i} \underset{\sim}{1}_{z} \tag{6.28}$$

where

$$\underset{\sim}{1}_{z} \equiv \underset{\sim}{\Omega}/|\underset{\sim}{\Omega}| \tag{6.29}$$

Taking the divergence of (6.28), and applying the equation of mass conservation, (6.2), we have

$$\frac{\partial}{\partial z}\{\rho u_{z}^{i}\} = -\frac{1}{2\Omega} \underset{\sim}{\nabla} \cdot \left\{(\underset{\sim}{F}_{B}^{i} + \underset{\sim}{F}) \times \underset{\sim}{1}_{z}\right\} \tag{6.30}$$

The z-component of equation (6.21") is

$$\frac{\partial}{\partial z} p_{1} = (\underset{\sim}{F}_{B}^{i} + \underset{\sim}{F}) \cdot \underset{\sim}{1}_{z} \tag{6.31}$$

Integrating equations (6.30) and (6.31) with respect to z, we obtain

$$\rho u_{z}^{i} = (\rho u_{z}^{i})_{b} - \frac{1}{2\Omega} \int_{z_b}^{z} \underset{\sim}{\nabla} \cdot \left\{(\underset{\sim}{F}_{B}^{i} + \underset{\sim}{F}) \times \underset{\sim}{1}_{z}\right\} dz \tag{6.32}$$

$$p_{1} = (p_{1})_{b} + \int_{z_b}^{z} (\underset{\sim}{F}_{B}^{i} + \underset{\sim}{F}) \cdot \underset{\sim}{1}_{z} \, dz \tag{6.33}$$

In equations (6.32) and (6.33), the quantities

$$(\rho u_z^i)_b = \rho u_z^i\{\varpi, \phi, z_b(\varpi,\phi)\} \qquad (6.34)$$

$$(p_1)_b = p_1\{\varpi, \phi, z_b(\varpi,\phi)\} \qquad (6.34')$$

are boundary values of ρu_z^i and p_1 . $(\rho u_z^i)_b$ must be determined by boundary layer analysis. In all these equations, z_b represents the value of z on the boundary for a particular pair of values (ϖ,ϕ) .

6.2.2 Boundary layer analysis

The boundary-layer equations (6.23)-(6.25) will have a consistent solution if

$$|(\underset{\sim}{B}^i)_b| / B_o^{(s)} = \mathcal{O}(1) \qquad (6.35)$$

$$|\underset{\sim}{B}_t^b| / B_o^{(s)} = \mathcal{O}(\delta/L) = \mathcal{O}(\varepsilon^{1/2}) \qquad (6.35')$$

$$|B_n^b| / B_o^{(s)} = \mathcal{O}(\delta^2/L^2) = \mathcal{O}(\varepsilon) \qquad (6.35")$$

and

$$|(\underset{\sim}{u}_t^i)_b| / u_o = \mathcal{O}(1) \qquad (6.36)$$

$$|(u_n^i)_b| / u_o = \mathcal{O}(\delta/L) = \mathcal{O}(\varepsilon^{1/2}) \qquad (6.36')$$

$$|\underset{\sim}{u}_t^b| / u_o = \mathcal{O}(1) \qquad (6.36")$$

$$|u_n^b| / u_o = \mathcal{O}(\delta/L) = \mathcal{O}(\varepsilon^{1/2}) \qquad (6.36''')$$

where δ is the thickness of the boundary layer and L is the length scale of variation of the "main flow" fields $(\underset{\sim}{u}^i, \underset{\sim}{B}^i)$. In these equations, $B_o^{(S)}$ is the average magnitude of the magnetic flux density on the boundary surface, S, and u_o is the average magnitude of the *tangential component* of $\underset{\sim}{u}^i$ on S. The parentheses $(\ )_b$ denote the *boundary value* of a "main flow" quantity such as $\underset{\sim}{u}^i$ or $\underset{\sim}{B}^i$, while the subscripts "n" and "t" denote the *normal* and *tangential* components of vectors near the boundary. The *tangential components* of $\underset{\sim}{u}$ and $\underset{\sim}{B}$ are defined as follows:

$$\underset{\sim}{u}_t \equiv \underset{\sim}{u} - u_n \underset{\sim}{1}_n \tag{6.37}$$

$$\underset{\sim}{B}_t \equiv \underset{\sim}{B} - B_n \underset{\sim}{1}_n \tag{6.37'}$$

If (6.35)-(6.36) are valid, *and the boundary of the conducting volume is spherical*, we may take components of the equations (6.23)-(6.25) in spherical polar coordinates $[r,\theta,\phi]$ to obtain (to zero order in δ/L)

$$\{2\rho\Omega\cos\theta\}\, u_\theta^b = \frac{1}{\mu}(B_n^i)_b \frac{\partial}{\partial n} B_\phi^b + \rho\nu \frac{\partial^2}{\partial n^2} u_\phi^b \tag{6.38}$$

$$-\{2\rho\Omega\cos\theta\}\, u_\phi^b = \frac{1}{\mu}(B_n^i)_b \frac{\partial}{\partial n} B_\theta^b + \rho\nu \frac{\partial^2}{\partial n^2} u_\theta^b \tag{6.39}$$

$$\frac{\partial^2}{\partial n^2} B_\theta^b = -\mu\sigma(B_n^i)_b \frac{\partial}{\partial n} u_\theta^b \tag{6.40}$$

$$\frac{\partial^2}{\partial n^2} B_\phi^b = -\mu\sigma(B_n^i)_b \frac{\partial}{\partial n} u_\phi^b \tag{6.41}$$

Equations (6.40) and (6.41) may be integrated, subject to the condition

$$\{\underset{\sim}{u}^b, \underset{\sim}{B}^b\} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \tag{6.42}$$

Carrying out the integration, we have

$$\frac{\partial}{\partial n} B_\theta^b = -\mu\sigma(B_n^i)_b u_\theta^b \tag{6.43}$$

$$\frac{\partial}{\partial n} B_\phi^b = -\mu\sigma(B_n^i)_b u_\phi^b \tag{6.44}$$

Substituting (6.43) and (6.44) into equations (6.38) and (6.39), and combining the equations, we obtain

$$\frac{\partial^2}{\partial n^2}(u_\phi^b + i u_\theta^b) = \frac{2\Omega}{\nu}\left\{\frac{\sigma}{2\rho\Omega}(B_n^i)_b^2 - i\cos\theta\right\}(u_\phi^b + i u_\theta^b) \tag{6.45}$$

Equation (6.45) has the solution

$$(u_\phi^b + i u_\theta^b) = -(u_\phi^i + i u_\theta^i)_b e^{-\gamma n} \tag{6.46}$$

satisfying the *no-slip* condition at the core-mantle interface, where

$$\gamma \equiv \sqrt{\frac{2\Omega}{\nu}\left\{\frac{\sigma}{2\rho\Omega}(B_n^i)^2 - i\cos\theta\right\}} \quad , \ \mathcal{R}e\,\gamma \geq 0 \tag{6.47}$$

Writing

$$\gamma = \mathcal{R}e\,\gamma + i\,\mathcal{I}m\,\gamma$$

we obtain the expressions

$$u_\theta^b = -\{(u_\theta^i)_b \cos(n\mathcal{I}m\gamma) + (u_\phi^i)_b \sin(n\mathcal{I}m\gamma)\} e^{-n\mathcal{R}e\gamma} \tag{6.48}$$

$$u_\phi^b = -\{(u_\phi^i)_b \cos(n\mathcal{I}m\gamma) - (u_\theta^i)_b \sin(n\mathcal{I}m\gamma)\} e^{-n\mathcal{R}e\gamma} \tag{6.48'}$$

$$B_\theta^b = \frac{\mu\sigma}{\mathcal{R}e\gamma}(B_n^i)_b \{(u_\theta^i)_b \cos(n\mathcal{I}m\gamma) + + (u_\phi^i)_b \sin(n\mathcal{I}m\gamma)\} e^{-n\mathcal{R}e\gamma} \tag{6.49}$$

$$B_\phi^b = \frac{\mu\sigma}{\mathcal{R}e\gamma}(B_n^i)_b \{(u_\phi^i)_b \cos(n\mathcal{I}m\gamma) + - (u_\theta^i)_b \sin(n\mathcal{I}m\gamma)\} e^{-n\mathcal{R}e\gamma} \tag{6.49'}$$

for the tangential components of the boundary-layer fields $(\underset{\sim}{u}^b, \underset{\sim}{B}^b)$. The normal components of these fields are still to be determined.

From the incompressibility condition (6.24),

$$\frac{\partial}{\partial n} u_n^b = -\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_t^b \tag{6.50}$$

Integrating this equation with respect to n , and making use of the condition (6.42), we obtain

$$u_n^b(0) = \int_0^\infty (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_t^b)\, dn \tag{6.51}$$

From equation (6.51) and the condition that u_n must vanish on the boundary, it follows that

$$(u_n^i)_b = -\int_0^\infty (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_t^b)\, dn = -\underset{\sim}{\nabla} \cdot \int_0^\infty \underset{\sim}{u}_t^b\, dn \tag{6.52}$$

Substituting (6.46) into equation (6.52), we have

$$(u_n^i)_b = \nabla \cdot \Big\{ \mathbf{1}_\theta \, \mathcal{R}e\Big[(u_\phi^i + i u_\theta^i)_b \int_0^\infty e^{-\gamma n}\, dn\Big] + \\ + \mathbf{1}_\phi \, \mathcal{I}m\Big[(u_\phi^i + i u_\theta^i)_b \int_0^\infty e^{-\gamma n}\, dn\Big] \Big\}$$

$$= \nabla \cdot \Big\{ \mathbf{1}_\theta \frac{1}{|\gamma|^2}\big[(u_\phi^i)_b \mathcal{R}e\,\gamma + (u_\theta^i)_b \mathcal{I}m\,\gamma\big] \\ + \mathbf{1}_\phi \frac{1}{|\gamma|^2}\big[(u_\theta^i)_b \mathcal{R}e\,\gamma - (u_\phi^i)_b \mathcal{I}m\,\gamma\big] \Big\} \qquad (6.53)$$

Therefore, from equations (6.53) and (6.28) the boundary value of ρu_z^i is given by

$$(\rho u_z^i)_b = \frac{1}{2\Omega}\Big\{(\nabla p_1)_b - (\mathbf{F}_B^i + \mathbf{F})_b\Big\} \cdot \mathbf{1}_\phi \tan\theta + \qquad (6.54) \\ + \rho \nabla \cdot \Big\{ \mathbf{1}_\theta \frac{1}{|\gamma|^2}\big[(u_\phi^i)_b \mathcal{R}e\,\gamma + (u_\theta^i)_b \mathcal{I}m\,\gamma\big] + \\ + \mathbf{1}_\phi \frac{1}{|\gamma|^2}\big[(u_\theta^i)_b \mathcal{R}e\,\gamma - (u_\phi^i)_b \mathcal{I}m\,\gamma\big] \Big\}$$

6.2.3 The full solution of the approximate Navier-Stokes equation

We are now in a position to obtain a complete solution to the *magnetogeostrophic equation* (6.21"), correct to **first** order in the boundary layer parameters. Substituting (6.32) and (6.54) into equation (6.28), we obtain

$$\mathbf{u}^i = \frac{1}{2\rho\Omega}\{\mathbf{f}_B^i + \mathbf{f}\} \qquad (6.55)$$

where

$$\underset{\sim}{f}_B^i \equiv \left\{ \underset{\sim}{F}_B^i - \underset{\sim}{\nabla} \int_{z_b}^{z} \underset{\sim}{F}_B^i \cdot \underset{\sim}{1}_z \, dz \right\} \times \underset{\sim}{1}_z + \tag{6.56}$$

$$+ \underset{\sim}{1}_z \left\{ (\underset{\sim}{F}_B^i)_b \cdot \underset{\sim}{1}_\phi \tan\theta + \underset{\sim}{1}_z \cdot \left[\underset{\sim}{\nabla} \times \int_{z_b}^{z} \underset{\sim}{F}_B^i \, dz \right] \right\} +$$

$$+ \underset{\sim}{1}_z \, 2\Omega \, \underset{\sim}{\nabla} \cdot \left\{ \underset{\sim}{1}_\theta \frac{1}{|\gamma|^2} \left[(u_\phi^i)_b \, \mathcal{R}e\,\gamma + (u_\theta^i)_b \, \mathcal{I}m\,\gamma \right] \right.$$

$$\left. + \underset{\sim}{1}_\phi \frac{1}{|\gamma|^2} \left[(u_\theta^i)_b \, \mathcal{R}e\,\gamma - (u_\phi^i)_b \, \mathcal{I}m\,\gamma \right] \right\}$$

$$\underset{\sim}{f} \equiv \left\{ \underset{\sim}{F} - \quad \nabla p_1 \quad - \underset{\sim}{\nabla} \int_{z_b}^{z} \underset{\sim}{F} \cdot \underset{\sim}{1}_z \, dz \right\} \times \underset{\sim}{1}_z + \tag{6.57}$$

$$- \underset{\sim}{1}_z \left\{ (\underset{\sim}{F})_b \cdot \underset{\sim}{1}_\phi \tan\theta - \underset{\sim}{\nabla}(p_1)_b \cdot \underset{\sim}{1}_\phi \tan\theta + \right.$$

$$\left. + \underset{\sim}{1}_z \cdot \left[\underset{\sim}{\nabla} \times \int_{z_b}^{z} \underset{\sim}{F} \, dz \right] \right\}$$

It should be noted that equation (6.55) is a *recursive* definition of $\underset{\sim}{u}^i$, since $\underset{\sim}{u}^i$ appears on both sides of the equation. However, the terms involving $\underset{\sim}{u}^i$ on the right hand side are of order $[\delta/L]$. The boundary layer thickness, δ , is given by

$$\delta \sim |\gamma|^{-1} \sim \sqrt{\frac{\nu}{2\Omega}} \left\{ \left(\frac{\sigma B_n^{i\,2}}{2\rho\Omega} \right)_b^2 + \cos^2\theta \right\}^{-1/4} \tag{6.58}$$

For the Earth's fluid core, we may take

$$\sigma \sim 3 \times 10^5 \text{ mho/m} \tag{6.59}$$

(see, for example, Gardiner and Stacey, 1971),

$$\rho \sim 10^4 \text{ kg/m}^3 \tag{6.60}$$

(see, for example, Jacobs, 1971d), and

$$(B_n^i)_b \sim 5 \text{ G} = 5 \times 10^{-4} \text{ T} \tag{6.61}$$

(see Table 2). Substituting these values into (6.58), along with the value $\Omega \sim 7 \times 10^{-5}$ sec^{-1}, we obtain a range of values for the boundary layer thickness as θ varies from 0 to $\pi/2$.

$$83\sqrt{\nu\,(\text{m}^2\cdot\text{sec}^{-1})} \lesssim \delta\,(\text{m.}) \lesssim 370\sqrt{\nu} \tag{6.62}$$

It should be noted that the upper limit on δ, obtained by setting $\theta = \pi/2$ in equation (6.58), is only valid if $(B_n^i)_b$ does not become small at this value of θ. If, on the other hand, the external magnetic field is an axial dipole, $(B_n^i)_b \propto \cos\theta$, and $\delta \to \infty$ as $\theta \to \pi/2$, in the first-order boundary layer approximation.

If the *Gans (1972a)* estimate of the kinematic viscosity in the Earth's fluid core is used *(see equation 6.15)*, (6.62) becomes

$$6.4 \text{ cm.} \lesssim \delta \lesssim 28 \text{ cm.} \tag{6.62'}$$

If, on the other hand, the *Hide (1971b)* estimate of the effective kinematic viscosity near the core-mantle interface is used *(see equation 6.15')*, (6.62) gives

$$830 \text{ m.} \lesssim \delta \lesssim 3700 \text{ m.} \tag{6.62''}$$

In both cases, $\delta \ll L$ when $L \sim r_o \sim 3 \times 10^6$ m, as expected.

6.3 Temporal behaviour of the external magnetic dipole moment

6.3.1 The boundary-layer control approximation

We are now in a position to examine the *boundary-layer control* equation (5.102) within the framework of the hydromagnetic dynamo problem. To first order in $[\delta/L]$, equations (6.45)-(6.47) give

$$\begin{aligned}(\nabla^2 \underset{\sim}{u})_b &\approx \left\{ \underset{\sim}{1}_\theta \frac{\partial^2}{\partial n^2} u^b_\theta + \underset{\sim}{1}_\phi \frac{\partial^2}{\partial n^2} u^b_\phi \right\}_{n=0} \\ &\approx \underset{\sim}{1}_\theta \left\{ \frac{2\Omega}{\nu} (u^i_\phi)_b \cos\theta - \frac{\sigma}{\rho\nu} (B^i_n)^2_b (u^i_\theta)_b \right\} \\ &\qquad - \underset{\sim}{1}_\phi \left\{ \frac{2\Omega}{\nu} (u^i_\theta)_b \cos\theta + \frac{\sigma}{\rho\nu} (B^i_n)^2_b (u^i_\phi)_b \right\}\end{aligned} \tag{6.63}$$

Similarly, from (6.55) we have, to first order,

$$\begin{aligned}(\underset{\sim}{u}^i)_b &\approx \frac{1}{2\rho\Omega} \Big\{ [\underset{\sim}{F}^i_B + \underset{\sim}{F} - \underset{\sim}{\nabla} p_1] \times \underset{\sim}{1}_z \\ &\qquad - \underset{\sim}{1}_z [\underset{\sim}{F}^i_B + \underset{\sim}{F} - \underset{\sim}{\nabla} p_1] \cdot \underset{\sim}{1}_\phi \tan\theta \Big\}_b \\ &\approx \frac{1}{2\rho\Omega} \left\{ \underset{\sim}{\mathfrak{F}} \times \underset{\sim}{1}_z - \underset{\sim}{1}_z \underset{\sim}{\mathfrak{F}} \cdot \underset{\sim}{1}_\phi \tan\theta \right\}\end{aligned} \tag{6.64}$$

where

$$\underset{\sim}{\mathfrak{F}} \equiv [\underset{\sim}{F}^i_B + \underset{\sim}{F} - \underset{\sim}{\nabla} p_1]_b \tag{6.65}$$

Combining equations (6.63) and (6.64), we obtain an expression for $(\nabla^2 \underset{\sim}{u})_b$ in terms of the boundary distributions of forces and magnetic fields.

$$(\nabla^2 \underset{\sim}{u})_b \approx -\frac{2\Omega}{\nu}\cos\theta\ \underset{\sim}{1}_r \times (\underset{\sim}{u}^i)_b - \frac{\sigma}{\rho\nu}(B_n^i)^2 (\underset{\sim}{u}^i)_b$$

$$\approx -\frac{1}{\rho\nu}\cos\theta\left\{\underset{\sim}{1}_r \times (\underset{\sim}{\mathcal{J}} \times \underset{\sim}{1}_z) - \underset{\sim}{1}_r \times \underset{\sim}{1}_z\, \mathcal{J}_\phi \tan\theta\right\}$$

$$-\frac{\sigma}{2\rho^2\Omega\nu}(B_n^i)_b^2\left\{\underset{\sim}{\mathcal{J}} \times \underset{\sim}{1}_z - \underset{\sim}{1}_z\, \mathcal{J}_\phi \tan\theta\right\}$$

$$\approx -\frac{1}{\rho\nu}\left\{\underset{\sim}{1}_\theta\, \mathcal{J}_\varpi \cos\theta + \underset{\sim}{1}_\phi\, \mathcal{J}_\phi\right\} \tag{6.66}$$

$$-\frac{\sigma}{2\rho^2\Omega\nu}(B_n^i)_b^2\left\{\underset{\sim}{1}_\theta\, \frac{\mathcal{J}_\phi}{\cos\theta} - \underset{\sim}{1}_\phi\, \mathcal{J}_\varpi\right\}$$

From the multipole expansion (5.66) of $\underset{\sim}{B}$ on the boundary of the conducting volume, and condition (6.35),

$$(B_n^i)_b \approx (\hat{B}_n)_b \approx \frac{\mu}{2\pi r_0^3}\underset{\sim}{T}^{(1)} \cdot \underset{\sim}{1}_r + \mathcal{O}\left\{\frac{\mu}{r_0^4}\left|\underset{\approx}{T}^{(2)}\right|\right\} \tag{6.67}$$

Substituting (6.66) and (6.67) into the *boundary-layer control* equation (5.102), with $\alpha = 0$, we obtain

$$\ddot{\underset{\sim}{T}}^{(1)} \approx \frac{3\eta}{4\pi r_0^3}\int_S (\nabla^2 \underset{\sim}{u})_b\ \underset{\sim}{1}_r \cdot \underset{\sim}{T}^{(1)}\, dS$$

$$\approx -\frac{3\eta}{4\pi r_0^3 \rho\nu}\underset{\sim}{T}^{(1)} \cdot \int_S \left\{\underset{\sim}{1}_r \underset{\sim}{1}_\theta\, \mathcal{J}_\varpi \cos\theta + \underset{\sim}{1}_r \underset{\sim}{1}_\phi\, \mathcal{J}_\phi\right\} dS +$$

$$-\frac{6\mu}{\rho^2\Omega\nu(4\pi r_0^3)^3}\left\{\underset{\sim}{T}^{(1)}\right\}^3 \cdot \int_S \underset{\sim}{1}_r^3\left\{\underset{\sim}{1}_\theta\, \frac{\mathcal{J}_\phi}{\cos\theta} - \underset{\sim}{1}_\phi\, \mathcal{J}_\varpi\right\} dS \tag{6.68}$$

6.3.2 The boundary force distribution in the geodynamo

In the geodynamo,

$$\underset{\sim}{\mathfrak{F}} = \left\{ \rho \underset{\sim}{g} - \underset{\sim}{\nabla} p + \tfrac{1}{2}\rho \underset{\sim}{\nabla}(\Omega^2\varpi^2) - \rho \dot{\underset{\sim}{\Omega}} \times \underset{\sim}{r} + \underset{\sim}{F}_B + \underset{\sim}{F}_{other} \right\}_b \tag{6.69}$$

where g is the acceleration due to gravity. Assuming *hydrostatic balance* in (6.69), to first order, we obtain

$$\underset{\sim}{\mathfrak{F}} = \left\{ \xi \rho \Omega^2 \varpi \underset{\sim}{1}_\varpi - \rho \dot{\underset{\sim}{\Omega}} \times \underset{\sim}{r} + \underset{\sim}{F}_B + \underset{\sim}{F}_{other} \right\}_b \tag{6.70}$$

where ξ represents the fraction of the centrifugal force which does not contribute to the hydrostatic balance. Taking components of (6.70) in cylindrical coordinates, we have

$$\mathfrak{F}_\varpi = \left\{ \xi \rho \Omega^2 \varpi + (\underset{\sim}{F}_B)_\varpi + (\underset{\sim}{F}_{other})_\varpi \right\}_b \tag{6.71}$$

$$\mathfrak{F}_\phi = \left\{ -\rho \dot{\Omega} \varpi + (\underset{\sim}{F}_B)_\phi + (\underset{\sim}{F}_{other})_\phi \right\}_b \tag{6.72}$$

We may now estimate the various terms in (6.71) and (6.72). The *Lorentz force* at the core-mantle interface, $(\underset{\sim}{F}_B)_b$, is given by

$$\left\{ (\underset{\sim}{F}_B)_\varpi \right\}_b = \left\{ j_\phi B_z - j_z B_\phi \right\}_b \tag{6.73}$$

$$\left\{ (\underset{\sim}{F}_B)_\phi \right\}_b = \left\{ j_z B_\varpi - j_\varpi B_z \right\}_b \tag{6.73'}$$

Assuming that the external medium is a nonconductor, we have

$$(\underset{\sim}{n} \cdot \underset{\sim}{j})_b = 0 \tag{6.74}$$

or, for a spherical boundary,

$$(j_\varpi)_b \sin\theta = \frac{1}{\mu}\left\{\frac{1}{\varpi}\frac{\partial}{\partial\phi}B_z - \frac{\partial}{\partial z}B_\phi\right\}_b \sin\theta \tag{6.74'}$$

$$= -(j_z)_b \cos\theta = \frac{1}{\mu}\left\{\frac{1}{\varpi}\frac{\partial}{\partial\phi}B_\varpi - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\phi)\right\}_b \cos\theta$$

Substituting (6.74') into (6.73'), we obtain

$$\{(\underset{\sim}{F}_B)_\phi\}_b = \frac{(j_z)_b}{\sin\theta}\{\underset{\sim}{B}\cdot\underset{\sim}{n}\}_b \tag{6.75}$$

where, from (6.74'),

$$(j_z)_b \sim \frac{1}{\mu}\left\{\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\phi)\right\}_b \tag{6.76}$$

Furthermore,

$$(j_\phi)_b = \frac{1}{\mu}\left\{\frac{\partial}{\partial z}B_\varpi - \frac{\partial}{\partial\varpi}B_z\right\}_b \tag{6.77}$$

As may be seen from *Table 2*, the toroidal magnetic field at the core-mantle interface is expected to be much smaller than the poloidal field. Within the framework of the *boundary-layer control approximation*, we may take

$$(B_\varpi)_b \sim (B_z)_b \sim B_0^{(s)} \sim 5\times10^{-4}\ \mathrm{T}. \tag{6.78}$$

$$(B_\phi)_b \sim 0 \tag{6.78'}$$

We may also take

$$\left(\frac{\partial B_\varpi}{\partial z}\right)_b \sim \left(\frac{\partial B_z}{\partial\varpi}\right)_b \sim B_0^{(s)}/L \sim 2\times10^{-10}\ \mathrm{T/m} \tag{6.79}$$

and

$$\left\{\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\phi)\right\}_b \sim \left\{\frac{\partial B_\phi}{\partial\varpi}\right\}_b \lesssim \frac{B_o}{L} \sim 3 \times 10^{-9}\ \mathrm{T/m.} \qquad (6.79')$$

where $B_o \sim 10^{-2}$ T is the magnitude of the field deep in the core. (6.79') may well be an overestimate, as it seems unlikely that the toroidal field increases linearly from its value at the core-mantle interface to its value deep in the core. Substituting (6.78)-(6.79') into (6.73) and (6.75), we have

$$\{(\underset{\sim}{F}_B)_\varpi\}_b \sim \{B_o^{(s)}\}^2/\mu L \sim 7 \times 10^{-8}\ \mathrm{nt/m^3} \qquad (6.80)$$

$$\{(\underset{\sim}{F}_B)_\phi\}_b \lesssim B_o B_o^{(s)}/\mu L \sim 1 \times 10^{-6}\ \mathrm{nt/m^3} \qquad (6.81)$$

The remaining terms in (6.70) may be evaluated using the estimates

$$\rho \sim 10^4\ \mathrm{kg/m^3}$$

$$L \sim 3 \times 10^6\ \mathrm{m} \qquad (6.82)$$

$$\Omega \sim 7 \times 10^{-5}\ \mathrm{rad/sec}$$

and

$$\dot{\Omega} \sim 2 \times 10^{-21}\ \mathrm{rad/sec^2} \qquad (6.83)$$

(see Allen, 1963, p. 109; Jacobs, 1970b). The estimate (6.83) for $\dot{\Omega}$ corresponds to the slow increase in the length of the day. Substituting these values into (6.71) and (6.72), we obtain

$$\rho\Omega^2\varpi \sim \rho\Omega^2 L \sim 150\ \mathrm{nt/m^3} \qquad (6.84)$$

$$\rho\dot{\Omega}\varpi \sim \rho\dot{\Omega} L \sim 6 \times 10^{-11}\ \mathrm{nt/m^3} \qquad (6.85)$$

The forces denoted by $\underset{\sim}{F}_{other}$ in equations (6.69)-(6.72) can only be estimated in specific models. We shall include the *precessional force term* defined in (1.68) as an example. From (1.68'),

$$\underset{\sim}{F}_P \approx \rho\left\{(\underset{\sim}{\Omega}'\times\underset{\sim}{\Omega})\times\underset{\sim}{r} - \frac{1}{2}\underset{\sim}{\nabla}\left[[(2\underset{\sim}{\Omega}+\underset{\sim}{\Omega}')\times\underset{\sim}{r}]\cdot[\underset{\sim}{\Omega}'\times\underset{\sim}{r}]\right]\right\} \qquad (6.86)$$

Assuming that the gradient term in (6.86), and a fraction $[1 - \frac{1}{\zeta}]$ of the term $\rho[(\underset{\sim}{\Omega}'\times\underset{\sim}{\Omega})\times\underset{\sim}{r}]$ are included in the hydrostatic balance in (6.69), we obtain

$$\underset{\sim}{F}_{other} \approx \frac{1}{\zeta}\rho\left\{(\underset{\sim}{\Omega}'\times\underset{\sim}{\Omega})\times\underset{\sim}{r}\right\} \qquad (6.87)$$

This term will have components in both the $\underset{\sim}{1}_{\varpi}$ and the $\underset{\sim}{1}_{\phi}$ directions. The magnitude of these contributions can be estimated as

$$|(\underset{\sim}{F}_{other})_b| \sim \frac{1}{\zeta_b}\rho\Omega(\Omega'\sin\chi)\,r_o \qquad (6.88)$$

where $[\Omega'\sin\chi]$ is the magnitude of the equatorial component of $\underset{\sim}{\Omega}'$. *Malkus (1971a)* gives the values

$$\Omega' \sim 7.71\times10^{-12}\ \text{rad/sec} \qquad (6.89)$$

$$\chi \sim 23.5° \qquad (6.89')$$

Using these values in (6.88), in conjunction with the values (6.82), we obtain

$$|(\underset{\sim}{F}_{other})_b| \sim \frac{1}{\zeta_b}\,8\times10^{-6}\ \text{nt/m}^3 \qquad (6.90)$$

6.3.3 Radial forces

We may now consider the effects of the ϖ-component of the boundary force distribution in the geodynamo on the temporal behaviour of the magnetic dipole moment. Ignoring the azimuthal component of $\underset{\sim}{\mathfrak{F}}$, we have from (6.68) that

$$\ddot{\underset{\sim}{T}}^{(1)} \approx -\frac{3\eta}{4\pi r_0^3 \rho \nu} \underset{\sim}{T}^{(1)} \cdot \int_S \underset{\sim}{1}_r \underset{\sim}{1}_\theta \mathfrak{F}_\varpi \cos\theta \, dS \qquad (6.91)$$

$$+ \frac{6\mu}{\rho^2 \Omega \nu (4\pi r_0^3)^3} \{\underset{\sim}{T}^{(1)}\}^3 \cdot \int_S \underset{\sim}{1}_r^3 \underset{\sim}{1}_\phi \mathfrak{F}_\varpi \, dS$$

From equations (6.71) and (6.88),

$$\mathfrak{F}_\varpi \approx \xi(\rho\Omega^2\varpi)_b + \{(\underset{\sim}{F}_B)_\varpi\}_b + $$

$$+ \frac{1}{\xi_b}\{\rho \underset{\sim}{1}_\varpi \cdot [(\underset{\sim}{\Omega}' \times \underset{\sim}{\Omega}) \times \underset{\sim}{r}]\}_b$$

$$\approx \xi(\rho\Omega^2\varpi)_b + \{(\underset{\sim}{F}_B)_\varpi\}_b - \frac{1}{\xi_b}\{\rho\varpi\Omega(\underset{\sim}{\Omega}' \cdot \underset{\sim}{1}_\varpi)\}_b \qquad (6.92)$$

The last term on the right hand side of (6.92) will not contribute to the first integral in (6.91), since

$$\int_S \varpi \underset{\sim}{1}_\varpi \underset{\sim}{1}_r \underset{\sim}{1}_\theta \cos\theta \, dS$$

$$= r_0 \int_S (\underset{\sim}{1}_x \cos\phi + \underset{\sim}{1}_y \sin\phi) \underset{\sim}{1}_r \underset{\sim}{1}_\theta \sin\theta\cos\theta \, dS$$

$$\equiv 0 \qquad (6.93)$$

as may be seen from detailed evaluation of the integrals, bearing in mind that

$$\int_S f(\theta,\phi)\, dS \equiv r_o^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} f(\theta,\phi)\, d\phi \tag{6.94}$$

After the ϕ-integration has been carried out in (6.93), the only terms remaining are proportional to

$$\{ \underset{\sim}{1}_x \underset{\sim}{1}_z \underset{\sim}{1}_x + \underset{\sim}{1}_y \underset{\sim}{1}_z \underset{\sim}{1}_y \} \sin^2\theta \cos^3\theta$$

and $$\{ \underset{\sim}{1}_x \underset{\sim}{1}_x \underset{\sim}{1}_z - \underset{\sim}{1}_y \underset{\sim}{1}_y \underset{\sim}{1}_z \} \sin^4\theta \cos\theta$$

When the θ-integration is carried out, both these terms vanish. Equation (6.92) may therefore be rewritten

$$\mathcal{F}_{\varpi}^{eff} \approx \xi(\rho\Omega^2\varpi)_b + \{(\underset{\sim}{F}_B)_{\varpi}\}_b \tag{6.95}$$

From the estimates (6.80) and (6.84), the *Lorentz force* term in (6.95) will be negligible provided that

$$\xi \gg 5 \times 10^{-10} \tag{6.96}$$

When (6.96) is valid, equation (6.91) becomes

$$\ddot{\underset{\sim}{I}}^{(1)} \approx -\frac{3\eta}{4\pi r_o^3 \rho\nu}\, \underset{\sim}{I}^{(1)} \cdot \int_S \underset{\sim}{1}_r \underset{\sim}{1}_\theta\, \xi\rho\Omega^2\varpi \cos\theta\, dS \tag{6.97}$$

$$+ \frac{6\mu}{\rho^2\Omega\nu(4\pi r_o^3)^3} \{\underset{\sim}{I}^{(1)}\}^3 \cdot \int_S \underset{\sim}{1}_r^3\, \underset{\sim}{1}_\phi\, \mathcal{F}_{\varpi}\, dS$$

or, expanding the first term,

$$\ddot{\underset{\sim}{T}}^{(1)} \approx -\frac{1}{5}\xi\frac{\eta\Omega^2}{\nu}\left\{T_x^{(1)}\underset{\sim}{1}_x + T_y^{(1)}\underset{\sim}{1}_y - 2T_z^{(1)}\underset{\sim}{1}_z\right\} + \tag{6.97'}$$

$$+ \frac{6\mu}{\rho^2\Omega\nu(4\pi r_o^3)^3}\left\{\underset{\sim}{T}^{(1)}\right\}^3 \cdot \int_S \underset{\sim}{1}_r^3\,\underset{\sim}{1}_\phi\,\mathcal{J}_\varpi\,dS$$

Taking components of (6.97'), we have

$$\ddot{T}_x^{(1)} \approx -\frac{1}{5}\xi\frac{\eta\Omega^2}{\nu}T_x^{(1)}\left\{1 + \frac{6\mu}{\eta\rho\Omega(4\pi r_o^3)^2}|\underset{\sim}{T}^{(1)}|^2\right\} \tag{6.98}$$

$$\ddot{T}_y^{(1)} \approx -\frac{1}{5}\xi\frac{\eta\Omega^2}{\nu}T_y^{(1)}\left\{1 + \frac{6\mu}{\eta\rho\Omega(4\pi r_o^3)^2}|\underset{\sim}{T}^{(1)}|^2\right\} \tag{6.98'}$$

$$\ddot{T}_z^{(1)} \approx \frac{2}{5}\xi\frac{\eta\Omega^2}{\nu}T_z^{(1)} \tag{6.98"}$$

The nonlinear terms in (6.98) and (6.98') have been obtained by evaluating the second integral in (6.97) and (6.97'). It can be shown that the *precessional force term* in (6.92) does not contribute to this integral, so that (6.95) is once again valid. As for the linear terms, the *Lorentz force* contribution has been ignored, subject to the assumption (6.96).

The *characteristic time* associated with variations governed by (6.98") is

$$\tau \sim \sqrt{\frac{5\nu}{2\xi\eta\Omega^2}} \tag{6.99}$$

For the geodynamo,

$$\tau \sim \sqrt{5/3\xi} \times 10^5 \text{ sec.} \tag{6.99'}$$

using the *Hide (1971b)* value (6.15') for the kinematic

viscosity at the core-mantle interface, the values (6.82), and the estimate

$$\eta \sim 3 \text{ m}^2\text{sec}^{-1} \tag{6.100}$$

for the magnetic diffusivity in the core.

It follows from equation (6.99') that *radial forces* can only account for dipole moment variations on a time scale of 10^3 years $\sim 3 \times 10^{10}$ seconds if

$$\xi \sim 2 \times 10^{-11} \tag{6.101}$$

However, this requirement is in direct contradiction to assumption (6.96). Thus there is no possibility of accounting for temporal variations of the geomagnetic dipole moment on scales $\sim 10^3$ years in terms of radial forces alone *unless the Lorentz force terms are included in equation (6.95).*

The Lorentz force terms in equation (6.95) can only be evaluated if the spatial behaviour of the *internal* magnetic field near the boundary is known. Inclusion of these terms must therefore destroy the "closed" nature of the *boundary layer control approximation*, and link the time dependence of the external potential field to the behaviour of the internal magnetic field, as well as to the body force density at the core-mantle interface. It follows that *if azimuthal forces at the core-mantle interface are neglected, temporal variations of the geomagnetic dipole moment on scales $\sim 10^3$ years cannot be accounted for in the "closed"*

boundary-layer control approximation.

6.3.4 Azimuthal forces, dipole wobble, and reversals

We shall now consider the effects of the ϕ-component of the boundary force distribution in the geodynamo on the temporal behaviour of the magnetic dipole moment. Ignoring the ϖ-component of $\underset{\sim}{\mathfrak{F}}$ in equation (6.68), we have

$$\ddot{\underset{\sim}{T}}^{(1)} \approx -\frac{3\eta}{4\pi r_0^3 \rho\nu} \underset{\sim}{T}^{(1)} \cdot \int_S \underset{\sim}{1}_r \underset{\sim}{1}_\phi \mathfrak{F}_\phi \, dS \; + \tag{6.102}$$

$$- \frac{6\mu}{\rho^2 \Omega \nu (4\pi r_0^3)^3} \{\underset{\sim}{T}^{(1)}\}^3 \cdot \int_S \underset{\sim}{1}_r^3 \underset{\sim}{1}_\theta \frac{\mathfrak{F}_\phi}{\cos\theta} \, dS$$

From equations (6.72) and (6.88),

$$\begin{aligned} \mathfrak{F}_\phi &\approx -(\rho\dot{\Omega}\varpi)_b + \{(\underset{\sim}{F}_B)_\phi\}_b + \frac{1}{\zeta_b}\{\rho \underset{\sim}{1}_\phi \cdot [(\underset{\sim}{\Omega}' \times \underset{\sim}{\Omega}) \times \underset{\sim}{r}]\}_b \\ &\approx -(\rho\dot{\Omega}\varpi)_b + \{(\underset{\sim}{F}_B)_\phi\}_b - \frac{1}{\zeta_b}\{\rho\Omega z (\underset{\sim}{\Omega}' \cdot \underset{\sim}{1}_\phi)\}_b \\ &\approx -(\rho\dot{\Omega}\varpi)_b + \{(\underset{\sim}{F}_B)_\phi\}_b + \\ &\qquad -\frac{1}{\zeta_b}\{\rho\Omega z(-\Omega_x' \sin\phi + \Omega_y' \cos\phi)\}_b \end{aligned} \tag{6.103}$$

In the *"closed" boundary layer approximation*, we must neglect the *Lorentz force* term, $\{(\underset{\sim}{F}_B)_\phi\}_b$, in equation (6.103). As may be seen from (6.81), (6.85), and (6.90), it is not clear that neglect of $\{(\underset{\sim}{F}_B)_\phi\}_b$ is a good approximation. However, if the toroidal magnetic field satisfies

the condition

$$\left|\left\{\frac{\partial \underset{\sim}{B}_\phi^{toroidal}}{\partial \varpi}\right\}_b\right| \ll \left|\left\{\frac{\mu\rho\dot{\Omega}\varpi}{(\underset{\sim}{B}\cdot\underset{\sim}{n})}\right\}_b\right| \sim 10^{-13}\ \mathrm{T/m} \qquad (6.104)$$

or if

$$\{(\underset{\sim}{F}_B)_\phi\}_b \propto P_{2m+1}(\cos\theta)\sin^{2n}\phi\cos^{2\ell}\phi \qquad (6.105)$$

so that $\{(\underset{\sim}{F}_B)_\phi\}_b$ makes no contribution to the integrals in equation (6.102), then the azimuthal *Lorentz force* term may be neglected in (6.103), and the *"closed" boundary layer approximation* is valid.

Substituting (6.103) into (6.102) and assuming that at least one of the conditions (6.104)-(6.105) is valid in the geodynamo, we obtain

$$\begin{aligned}\ddot{T}_x^{(1)} \approx & -\frac{\eta\dot{\Omega}}{\nu}T_y^{(1)} + \frac{\eta\Omega\Omega_x'}{2\zeta_b\nu}T_z^{(1)} + \\ & + \frac{6\mu\dot{\Omega}}{5\rho\Omega\nu(4\pi r_o^3)^2}|\underset{\sim}{T}^{(1)}|^2 T_x^{(1)} + \\ & + \frac{3\mu}{10\zeta_b\rho\nu(4\pi r_o^3)^2}T_z^{(1)}\Big\{\Omega_y'(3T_x^{(1)2} + T_y^{(1)2} + 2T_z^{(1)2}) + \\ & \qquad - 2\Omega_x' T_x^{(1)} T_y^{(1)}\Big\}\end{aligned} \qquad (6.106)$$

$$\begin{aligned}\ddot{T}_y^{(1)} \approx & \frac{\eta\dot{\Omega}}{\nu}T_x^{(1)} + \frac{\eta\Omega\Omega_y'}{2\zeta_b\nu}T_z^{(1)} + \\ & + \frac{6\mu\dot{\Omega}}{5\rho\Omega\nu(4\pi r_o^3)^2}|\underset{\sim}{T}^{(1)}|^2 T_y^{(1)} + \\ & - \frac{3\mu}{10\zeta_b\rho\nu(4\pi r_o^3)^2}T_z^{(1)}\Big\{\Omega_x'(T_x^{(1)2} + 3T_y^{(1)2} + 2T_z^{(1)2}) + \\ & \qquad - 2\Omega_y' T_x^{(1)} T_y^{(1)}\Big\}\end{aligned} \qquad (6.106')$$

$$\ddot{T}_z^{(1)} \approx -\frac{4\mu\dot{\Omega}}{5\rho\Omega\nu(4\pi r_o^3)^2} T_z^{(1)}\left\{2\left(T_x^{(1)2}+T_y^{(1)2}\right)+T_z^{(1)2}\right\}+$$

$$+\frac{6\mu}{5\zeta_b\rho\nu(4\pi r_o^3)^2}|\underset{\sim}{T}^{(1)}|^2\left\{\Omega'_x T_y^{(1)}-\Omega'_y T_x^{(1)}\right\} \qquad (6.106")$$

on carrying out the integrations.

When the nonlinear terms are neglected in (6.106) and (6.106'), the equations can be written in the form

$$\ddot{T}_{eq}^{(1)} \approx i\frac{\eta\dot{\Omega}}{\nu}T_{eq}^{(1)}+\frac{\eta\Omega\Omega'}{2\nu\zeta_b}T_z^{(1)}e^{-i\Omega' t}\sin\chi \qquad (6.107)$$

where

$$T_{eq}^{(1)} \equiv T_x^{(1)}+iT_y^{(1)} \qquad (6.108)$$

In equation (6.107) we have assumed that the equatorial component of the Earth's precession can be represented by

$$\Omega'_x+i\Omega'_y = (\Omega'\sin\chi)e^{-i\Omega' t} \qquad (6.109)$$

where Ω' and χ are the quantities defined in (6.89) and (6.89').

The general solution of equation (6.108) is

$$T_{eq}^{(1)} = A_1 e^{(1-i)\sqrt{\frac{\eta|\dot{\Omega}|}{2\nu}}\,t}+A_2 e^{-(1-i)\sqrt{\frac{\eta|\dot{\Omega}|}{2\nu}}\,t}+$$

$$-\frac{\eta\Omega\Omega'\sin\chi\left\{\Omega'^2+i\,\eta|\dot{\Omega}|/\nu\right\}}{2\left\{\Omega'^4+\eta^2|\dot{\Omega}|^2/\nu^2\right\}\nu\zeta_b}T_z^{(1)}e^{-i\Omega' t} \qquad (6.110)$$

where A_1 and A_2 are complex constants. Thus a typical

solution will be of the form

$$T_x^{(1)} = \{A_1' \cos\gamma' t - A_2' \sin\gamma' t\} e^{-\gamma' t} + \tag{6.111}$$

$$- \frac{\eta\Omega\Omega' \sin\chi}{2\nu\zeta_b\{\Omega'^4 + 4\gamma'^4\}} T_z^{(1)} \{\Omega'^2 \cos\Omega' t + 2\gamma'^2 \sin\Omega' t\}$$

$$T_y^{(1)} = \{A_1' \sin\gamma' t + A_2' \cos\gamma' t\} e^{-\gamma' t} + \tag{6.111'}$$

$$+ \frac{\eta\Omega\Omega' \sin\chi}{2\nu\zeta_b\{\Omega'^4 + 4\gamma'^4\}} T_z^{(1)} \{\Omega'^2 \sin\Omega' t - 2\gamma'^2 \cos\Omega' t\}$$

where A_1' and A_2' are real constants, and

$$\gamma' \equiv \sqrt{\eta|\dot{\Omega}|/2\nu} \tag{6.112}$$

In deriving (6.111) and (6.111') from (6.110), we have retained only those terms which represent *precession* of the equatorial dipole moment with angular frequency Ω', and *eastward drift* with angular frequency γ'. (It should be noted that in (6.110) we have assumed that $\dot{\Omega} < 0$, as is the case in the geodynamo.) From equation (6.112), the period of eastward drift is given by

$$T_{e.d.} = 2\pi/\gamma' = 2\pi\sqrt{2\nu/\eta|\dot{\Omega}|} \tag{6.113}$$

Substituting the estimates (6.83) and (6.100) into (6.113) we obtain the period

$$T_{e.d.} \sim (3{\cdot}8 \times 10^3)\sqrt{\nu} \quad (m^2 sec^{-1})^{-1/2} \text{ years} \tag{6.113'}$$

for the geodynamo.

Although the geomagnetic dipole axis has apparently drifted in a *westward* direction since 1700 A.D. *(Kawai and Hirooka, 1967)*, prior to 1600 A.D. the drift was apparently eastward *(Kawai and Hirooka, 1967; Márton, 1970)*. Several estimates have been made of the period of this eastward drift. For example, *Kawai and Hirooka (1967)* obtain the period $T_{e.d.} \sim 1500$ years, while *Márton (1970)* obtains $T_{e.d.} \sim 1800$ years. *Pudovkin and Valuyeva (1967, 1972)*, using an eccentric dipole model with an elliptical drift trajectory, obtain the period $T \sim 1200$ years (it would be inappropriate to refer to this model as one of "eastward drift"). Substituting these estimates into (6.113'), we obtain estimates of the kinematic viscosity at the core-mantle interface.

$$T = 1200 \text{ years} \rightarrow \nu \sim 1.0 \text{ m}^2/\text{sec} \tag{6.114}$$

$$T = 1500 \text{ years} \rightarrow \nu \sim 1.6 \text{ m}^2/\text{sec} \tag{6.114'}$$

$$T = 1800 \text{ years} \rightarrow \nu \sim 2.3 \text{ m}^2/\text{sec} \tag{6.114"}$$

These values are consistent with the estimate of *Hide (1971b)* that the kinematic viscosity at the core-mantle interface is $\lesssim 10^2 \text{ m}^2/\text{sec}$. From (6.62) we see that the estimates of ν given in (6.114)-(6.114") imply that the *boundary layer thickness* at the core-mantle interface lies in the range $100 \text{ m} \lesssim \delta \lesssim 600 \text{ m}$, giving good agreement with Hide's estimate $\delta < 1$ km .

In obtaining equations (6.111) and (6.111') from equation (6.110), we have ignored the terms which lead to *westward drift* with the angular frequency γ'. However, these terms may well be important for explaining the observed behaviour of the geomagnetic dipole. The data presented by *Kawai and Hirooka (1967)* appear to indicate that the amplitude of the equatorial dipole *increases* during periods of westward drift. It may be seen from equation (6.110) that westward drift with angular frequency γ' is associated with *growing* solutions, whereas eastward drift is associated with *decaying* solutions. A combination of the two types of solution might well account for the behaviour described by *Kawai and Hirooka*.

Westward drift also arises from the second group of terms in (6.111) and (6.111'). These terms are associated with the *precession* of the Earth's axis of rotation at the angular frequency Ω', corresponding to a period

$$T_{pr} \sim 25{,}800 \text{ years} \tag{6.115}$$

(Malkus, 1971a). In order to ensure that these terms do not dominate the terms described in the last paragraph, we must require that

$$\frac{\eta\Omega\Omega'\sin\chi}{2\nu\zeta_b\sqrt{\Omega'^4+4\gamma'^4}} \ll \frac{|T_{eq}^{(1)}|}{|T_z^{(1)}|} \sim \tan\{11.5°\} \sim 0.2 \tag{6.116}$$

or, making use of the values (6.82), (6.83), (6.89), (6.89') and (6.100), and the value of ν given in (6.114),

$$\frac{1}{\zeta_b} \ll 3 \times 10^{-6} \tag{6.116'}$$

Thus nearly all the precessional torque on the core fluid at the core-mantle interface must be balanced. It is interesting to note that if the value of ζ used by *Malkus (1971a)* for the flow in the main body of the core ($\zeta = 4$) is used in place of ζ_b in (6.116), the inequality can only be satisfied if $\nu \gg 7 \times 10^6\ m^2/sec$.

The effect of the nonlinear terms in (6.106) and (6.106') can be assessed fairly simply if the dipole moment is predominantly axial. Under these circumstances, the magnitude of $\underset{\sim}{T}^{(1)}$ is approximately independent of $T_x^{(1)}$ and $T_y^{(1)}$. If (6.116') is valid, only the first nonlinear term in each equation need be considered. The effect of these terms will be to introduce a long-period modulation in the solution given in (6.110). The period of this modulation can be estimated as

$$T_{mod} \sim 4\sqrt{2}\,\pi \left\{\eta|\dot{\Omega}|/\nu\right\}^{1/2} \left\{\frac{10\rho\Omega\nu\mu}{3|\dot{\Omega}|B_n^2}\right\}$$

$$\sim 8 \times 10^{12}\ \text{sec.} \sim 3 \times 10^5\ \text{years} \tag{6.117}$$

There will also be growth and/or decay on this time scale.

If the dipole moment is predominantly axial, equation (6.106"), which describes the behaviour of $T_z^{(1)}$, may be rewritten approximately as

$$\ddot{T}_z^{(1)} \approx P\, T_z^{(1)\,3} \tag{6.118}$$

where

$$P \equiv \frac{4\mu\,|\dot{\Omega}|}{5\rho\Omega\nu(4\pi r_o^3)^2} \tag{6.118'}$$

Carrying out the first integration, we obtain

$$\{\dot{T}_z^{(1)}\}^2 - \{\dot{T}_z^{(1)}\}_o^2 = \tfrac{1}{2}P\{[T_z^{(1)}]^4 - [T_z^{(1)}]_o^4\} \tag{6.119}$$

indicating that the axial dipole moment varies on time scales of the order

$$\tau \sim \frac{1}{|T_z^{(1)}|}\sqrt{P/2} \sim \frac{1}{|T_z^{(1)}|}\sqrt{\frac{5\rho\Omega\nu(4\pi r_o^3)^2}{4\mu|\dot{\Omega}|}}$$

$$\sim \frac{1}{|B_n|}\sqrt{\frac{10\rho\Omega\mu\nu}{|\dot{\Omega}|}}$$

$$\sim 1.3 \times 10^{11} \text{ seconds} \sim 4 \times 10^3 \text{ years} \tag{6.120}$$

As may be seen from *Table 13*, the time scale given in (6.120) is similar to that characterizing the behaviour of the axial dipole moment during a reversal.

If, on the other hand, the axial dipole moment is small compared with the equatorial dipole moment, equation (6.106") is approximately linear in $T_z^{(1)}$. The solution to the equation will be of the form

$$T_z^{(1)} \approx a_1' e^{t/\tau_1} + a_2' e^{-t/\tau_1} \tag{6.121}$$

where

$$\tau_1 = \sqrt{\frac{5\rho\Omega\nu\mu}{2B_n^2|\dot{\Omega}|}} \sim |B_n|^{-1}\ T\cdot\text{year} \tag{6.121'}$$

where $|B_n|$ represents the typical magnitude of the predominantly equatorial dipole field at the core-mantle interface. If $|B_n|$ lies in the range

$$1 \times 10^{-4}\, T \lesssim |B_n| \lesssim 5 \times 10^{-4}\, T \tag{6.122}$$

then

$$1 \times 10^{4}\, y. \gtrsim \tau_1 \gtrsim 2 \times 10^{3}\, y. \tag{6.123}$$

and the time scale of variation of the axial dipole moment will be of the same order of magnitude as that estimated in (6.120).

Summarizing the results of this section, we may state that the azimuthal force term $-\rho(\dot{\underset{\sim}{\Omega}} \times \underset{\sim}{r})$ has a considerable influence on the temporal behaviour of the geomagnetic dipole moment. The linear terms in the equations for the equatorial components of the dipole moment account for the *dipole wobble* reported by *Kawai and Hirooka (1967)* if the kinematic viscosity at the core-mantle interface is of the order $\nu \sim 1\ m^2/sec$. When this value of ν is assumed, the nonlinear terms in the equations for the equatorial geomagnetic dipole give rise to modulations on the time scale 3×10^5 years - which is of roughly the same order as the observed interval between polarity reversals during the last 50 m.y. Finally, when the value of ν implied by the dipole wobble is used, the

equation for the axial dipole moment indicates that the axial dipole can vary on time scales similar to those which characterize *polarity transitions*. It appears, therefore, that a detailed examination of simultaneous solutions to equations (6.106)-(6.106"), with the *precessional force* terms omitted, may well lead to a better understanding of dipole moment variations. However, before any great importance can be attached to variations predicted by these equations on time scales $\gg 10^4$ years, it will be necessary to include estimates of dissipative terms, as was pointed out in *section 5.5.1*.

We shall not carry our investigation of the *boundary layer control approximation* any further in this thesis.

6.4 Effects of an inhomogeneous turbulent force distribution outside the boundary layer in the geodynamo

6.4.1 The fluctuating dynamo equations

In the *mean field electrodynamic* approach, the dynamo equations (6.1)-(6.4) become

$$\{\partial/\partial t - \eta\nabla^2\}\mathbf{B}' = \mathrm{curl}\{\mathbf{u}'\times\overline{\mathbf{B}} + \overline{\mathbf{u}}\times\mathbf{B}' + \mathbf{u}'\times\mathbf{B}' - \overline{\mathbf{u}'\times\mathbf{B}'}\} \tag{6.124}$$

$$\nabla\cdot\mathbf{B}' = 0 \tag{6.125}$$

$$\{\partial/\partial t - \nu\nabla^2\}\mathbf{u}' + \{\overline{\mathbf{u}}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\overline{\mathbf{u}} + \mathbf{u}'\cdot\nabla\mathbf{u}' - \overline{\mathbf{u}'\cdot\nabla\mathbf{u}'}\} + 2\mathbf{\Omega}\times\mathbf{u}' \tag{6.126}$$

$$= \frac{1}{\rho}\{\mathbf{F}' - \nabla p'\} + \frac{1}{\rho\mu}\{(\nabla\times\mathbf{B}')\times\overline{\mathbf{B}} + (\nabla\times\overline{\mathbf{B}})\times\mathbf{B}' + (\nabla\times\mathbf{B}')\times\mathbf{B}' - \overline{(\nabla\times\mathbf{B}')\times\mathbf{B}'}\}$$

$$\nabla\cdot\mathbf{u}' = -\{\nabla\rho/\rho\}\cdot\mathbf{u}' \tag{6.127}$$

$$\{\partial/\partial t - \eta\nabla^2\}\overline{\mathbf{B}} = \mathrm{curl}\{\overline{\mathbf{u}}\times\overline{\mathbf{B}} + \overline{\mathbf{u}'\times\mathbf{B}'}\} \tag{6.128}$$

$$\nabla\cdot\overline{\mathbf{B}} = 0 \tag{6.129}$$

$$\{\partial/\partial t - \eta\nabla^2\}\overline{\mathbf{u}} + \{\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}} + \overline{\mathbf{u}'\cdot\nabla\mathbf{u}'}\} + 2\mathbf{\Omega}\times\overline{\mathbf{u}} =$$

$$= \frac{1}{\rho}\{\overline{\mathbf{F}} - \nabla\overline{p}\} + \frac{1}{\rho\mu}\{(\nabla\times\overline{\mathbf{B}})\times\overline{\mathbf{B}} + \overline{(\nabla\times\mathbf{B}')\times\mathbf{B}'}\} \tag{6.130}$$

$$\nabla\cdot\overline{\mathbf{u}} = -\{\nabla\rho/\rho\}\cdot\overline{\mathbf{u}} \tag{6.131}$$

In the *hydromagnetic first order smoothing approximation*, the fluctuating dynamo equations (6.124)-(6.127) become

$$\{\partial/\partial t - \eta\nabla^2\}\underset{\sim}{B}' = \overline{\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}' - \overline{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}\underset{\sim}{B}' - \underset{\sim}{u}'\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{B}} + \underset{\sim}{B}'\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{u}} \tag{6.132}$$

$$\{\partial/\partial t - \nu\nabla^2\}\underset{\sim}{u}' + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}' = -\overline{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}' - \underset{\sim}{u}'\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{u}} + \tag{6.133}$$

$$+\frac{1}{\rho}\left\{\underset{\sim}{F}' - \underset{\sim}{\nabla}P' + \frac{1}{\mu}\left[(\underset{\sim}{\nabla}\times\underset{\sim}{B}')\times\overline{\underset{\sim}{B}} + (\underset{\sim}{\nabla}\times\overline{\underset{\sim}{B}})\times\underset{\sim}{B}'\right]\right\}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{B}' = 0 \tag{6.134}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}' = -\{\underset{\sim}{\nabla}\rho/\rho\}\cdot\underset{\sim}{u}' \tag{6.135}$$

It should be noted that in these equations the density is not assumed to have a fluctuating component.

6.4.2 The first-order solution of the fluctuating dynamo equations

Making use of the formalism developed in *Chapter 4*, and the *Fourier-Stieltjes transforms* (4.7a,b), we may rewrite equations (6.132)-(6.135) in the form

$$
\begin{aligned}
&(i\omega+\eta k^2)\underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\} - 2i\eta\underset{\sim}{k}\cdot\underset{\sim}{\nabla}\underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\} + \\
&\qquad -\eta\nabla^2\underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\} \\
&= i(\underset{\sim}{k}\cdot\underset{\sim}{\overline{B}})\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} + \underset{\sim}{\overline{B}}\cdot\underset{\sim}{\nabla}\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} + \\
&\quad - i(\underset{\sim}{k}\cdot\underset{\sim}{\overline{u}})\underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\} - \underset{\sim}{\overline{u}}\cdot\nabla\underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\} + \\
&\quad - \{\underset{\approx}{U}\cdot[d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots]\}\cdot\underset{\sim}{\nabla}\underset{\sim}{\overline{B}} + \{\underset{\approx}{\beta}\cdot[d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots]\}\cdot\nabla\underset{\sim}{\overline{u}}
\end{aligned}
\tag{6.136}
$$

$$
\begin{aligned}
&(i\omega+\nu k^2)\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} + 2\underset{\sim}{\Omega}\times\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} + \\
&\qquad - 2i\nu\underset{\sim}{k}\cdot\underset{\sim}{\nabla}\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} - \nu\nabla^2\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} = \\
&= \frac{1}{\rho}\{\mathcal{F}d\underset{\sim}{f} - i\underset{\sim}{k}P\,dp - \underset{\sim}{\nabla}P\,dp\} + \\
&\quad + \frac{1}{\rho\mu}\{[i\underset{\sim}{k}\times\underset{\approx}{\beta}\cdot(d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots) + \underset{\sim}{\nabla}\times\underset{\approx}{\beta}\cdot(d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots)]\times\underset{\sim}{\overline{B}} + \\
&\qquad\qquad + (\underset{\sim}{\nabla}\times\underset{\sim}{\overline{B}})\times\underset{\approx}{\beta}\cdot(d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots)\} + \\
&\quad - i(\underset{\sim}{k}\cdot\underset{\sim}{\overline{u}})\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} - (\underset{\sim}{\overline{u}}\cdot\underset{\sim}{\nabla})\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} + \\
&\quad - \{\underset{\approx}{U}\cdot[d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots]\}\cdot\underset{\sim}{\nabla}\underset{\sim}{\overline{u}}
\end{aligned}
\tag{6.137}
$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} + \underset{\sim}{\nabla}\cdot\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} =$$

$$= -\{\underset{\sim}{\nabla}\rho/\rho\}\cdot\underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\} \qquad (6.138)$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\} = -\underset{\sim}{\nabla}\cdot\underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\} \qquad (6.139)$$

where

$$\underset{\sim}{u}'(\underset{\sim}{x},t) = \iint_{\underset{\sim}{k}\omega} \underset{\approx}{U}\cdot\{d\underset{\sim}{Z}^{(0)}+d\underset{\sim}{Z}^{(1)}+\dots\}e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t\}} \qquad (6.140)$$

$$\underset{\sim}{B}'(\underset{\sim}{x},t) = \iint_{\underset{\sim}{k}\omega} \underset{\approx}{\beta}\cdot\{d\underset{\sim}{Y}^{(0)}+d\underset{\sim}{Y}^{(1)}+\dots\}e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t\}} \qquad (6.140')$$

$$\underset{\sim}{F}'(\underset{\sim}{x},t) = \iint_{\underset{\sim}{k}\omega} \mathcal{F}\, d\underset{\sim}{f}(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t\}} \qquad (6.140'')$$

$$p'(\underset{\sim}{x},t) = \iint_{\underset{\sim}{k}\omega} P\, dp(\underset{\sim}{k},\omega)\, e^{i\{\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t\}} \qquad (6.140''')$$

In dealing with these equations, we shall assume that the mean flow $\bar{\underset{\sim}{u}}$ is small enough in magnitude for the terms $\bar{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}'$ and $\bar{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{B}'$ to be treated as second-order quantities.

The first-order equations are

$$(i\omega+\eta k^2)\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} = i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{B}})\,\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} \qquad (6.141)$$

$$(i\omega+\nu k^2)\,\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} + 2\underset{\sim}{\Omega}\times\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} = \qquad (6.142)$$

$$= \frac{1}{\rho}\{\mathcal{F}\,d\underset{\sim}{f} - i\underset{\sim}{k}\,P\,dp\} + \frac{1}{\rho\mu}\{i\underset{\sim}{k}\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\}\times\bar{\underset{\sim}{B}}$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} = 0 = i\underset{\sim}{k}\cdot\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} \qquad (6.143)$$

Solving for $\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}$ and $\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}$ under the assumption

$$\underset{\sim}{k}\cdot d\underset{\sim}{f}(\underset{\sim}{k},\omega) = 0 \tag{6.144}$$

we obtain

$$\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} = \frac{i(\underset{\sim}{k}\cdot\overline{\underset{\approx}{B}})}{\eta k^2+i\omega}\,\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} \tag{6.145}$$

$$\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} = \frac{\mathcal{F}}{\rho D}\left\{\sigma k^2 d\underset{\sim}{f} - 2(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})\,\underset{\sim}{k}\times d\underset{\sim}{f}\right\} \tag{6.146}$$

where

$$\sigma \equiv \nu k^2+i\omega + \frac{(\underset{\sim}{k}\cdot\overline{\underset{\approx}{B}})^2}{\rho\mu\{\eta k^2+i\omega\}} \tag{6.147}$$

$$D \equiv \sigma^2 k^2 + 4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^2 \tag{6.148}$$

In these equations we have adopted a notation closely similar to that used by *Moffatt (1972)*.

6.4.3 Helicity and $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ for a locally isotropic force distribution

In a study of turbulent dynamo action, the principal quantities of interest are the *turbulent intensity*, the *helicity*, and the *fluctuating e.m.f.* These quantities are given by the expressions

$$\overline{\underset{\sim}{u}' \cdot \underset{\sim}{u}'} = \iint_{\underset{\sim}{k}\omega} \overline{(\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^*) \cdot (\underset{\approx}{U} \cdot d\underset{\sim}{Z})}$$

$$= \iint_{\underset{\sim}{k}\omega} \overline{(\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}) \cdot (\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)})} + \ldots \qquad (6.149)$$

$$\overline{\underset{\sim}{u}' \cdot \mathrm{curl}\, \underset{\sim}{u}'} = \iint_{\underset{\sim}{k}\omega} \overline{(\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^*) \cdot \{ i\underset{\sim}{k} \times \underset{\approx}{U} \cdot d\underset{\sim}{Z} + \underset{\sim}{\nabla} \times \underset{\approx}{U} \cdot d\underset{\sim}{Z} \}}$$

$$= \iint_{\underset{\sim}{k}\omega} \overline{\{ \underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*} \} \cdot \{ i\underset{\sim}{k} \times \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)} \}} +$$

$$+ \iint_{\underset{\sim}{k}\omega} \{ \overline{[\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(1)*}] \cdot [i\underset{\sim}{k} \times \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}]} + \qquad (6.150)$$

$$+ \overline{[U^* \cdot dZ^{(0)*}] \cdot [i\underset{\sim}{k} \times \underset{\sim}{U} \cdot d\underset{\sim}{Z}^{(1)} + \underset{\sim}{\nabla} \times \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}]} \}$$

$$+ \ldots$$

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'} = \mathcal{R}e \iint_{\underset{\sim}{k}\omega} \overline{(\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^*) \times (\underset{\approx}{\beta} \cdot d\underset{\sim}{Y})}$$

$$= \mathcal{R}e \iint_{\underset{\sim}{k}\omega} \overline{[\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}] \times [\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(0)}]} + \qquad (6.151)$$

$$+ \mathcal{R}e \iint_{\underset{\sim}{k}\omega} \{ \overline{[\underset{\sim}{U}^* \cdot d\underset{\sim}{Z}^{(1)*}] \times [\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(0)}]} + \overline{[\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}] \times [\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(1)}]} \}$$

The first-order terms in the integrands in equations (6.149)-(6.151) can be rewritten, making use of equations (6.145) and (6.146).

$$\overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{\mathcal{Z}}^{(0)*})\cdot(\underset{\approx}{U}\cdot d\underset{\sim}{\mathcal{Z}}^{(0)})} = \tag{6.152}$$

$$= \left|\frac{\mathcal{Z}}{\rho D}\right|^{2}\Big\{ |\sigma|^{2}k^{4}\,\overline{d\underset{\sim}{f}^{*}\cdot d\underset{\sim}{f}} + 4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^{2}\,\overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\cdot(\underset{\sim}{k}\times d\underset{\sim}{f})} + \\ - 2k^{2}(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})\big[\sigma^{*}\,\overline{d\underset{\sim}{f}^{*}\cdot(\underset{\sim}{k}\times d\underset{\sim}{f})} + \sigma\,\overline{d\underset{\sim}{f}\cdot(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})}\big]\Big\}$$

$$\overline{\{\underset{\approx}{U}^{*}\cdot d\underset{\sim}{\mathcal{Z}}^{(0)*}\}\cdot\{i\underset{\sim}{k}\times\underset{\approx}{U}\cdot d\underset{\sim}{\mathcal{Z}}^{(0)}\}} = \tag{6.153}$$

$$= i\left|\frac{\mathcal{Z}}{\rho D}\right|^{2}\Big\{ |\sigma|^{2}k^{4}\,\overline{d\underset{\sim}{f}^{*}\cdot(\underset{\sim}{k}\times d\underset{\sim}{f})} + 4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^{2}\,\overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\cdot[\underset{\sim}{k}\times(\underset{\sim}{k}\times d\underset{\sim}{f})]} + \\ - 2k^{2}(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})\big[\sigma^{*}\,\overline{d\underset{\sim}{f}^{*}\cdot(\underset{\sim}{k}\times\{\underset{\sim}{k}\times d\underset{\sim}{f}\})} + \sigma\,\overline{(\underset{\sim}{k}\times d\underset{\sim}{f})\cdot(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})}\big]\Big\}$$

$$\overline{\{\underset{\approx}{U}^{*}\cdot d\underset{\sim}{\mathcal{Z}}^{(0)*}\}\times\{\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\}} = \tag{6.154}$$

$$= \frac{i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{B}})}{i\omega+\eta k^{2}}\left|\frac{\mathcal{Z}}{\rho D}\right|^{2}\Big\{ |\sigma|^{2}k^{4}\,\overline{d\underset{\sim}{f}^{*}\times d\underset{\sim}{f}} + 4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^{2}\,\overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\times(\underset{\sim}{k}\times d\underset{\sim}{f})} + \\ - 2k^{2}(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})\big[\sigma^{*}\,\overline{d\underset{\sim}{f}^{*}\times(\underset{\sim}{k}\times d\underset{\sim}{f})} + \sigma\,\overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\times d\underset{\sim}{f}}\big]\Big\}$$

We shall assume that the turbulent force distribution is *locally isotropic*, so that

$$\overline{df_i^{*}\,df_j} = \frac{\psi(k,\omega)}{4\pi k^{4}}\left\{k^{2}\delta_{ij} - k_i k_j\right\} \tag{6.155}$$

Under this assumption, $d\underset{\sim}{f}$ and $d\underset{\sim}{f}^{*}$ satisfy the equations:

$$\overline{d\underset{\sim}{f}^{*}\cdot d\underset{\sim}{f}} \quad = \quad \psi/2\pi k^2 \qquad (6.156a)$$

$$\overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\cdot(\underset{\sim}{k}\times d\underset{\sim}{f})} \quad = \quad \psi/2\pi \qquad (6.156b)$$

$$\overline{d\underset{\sim}{f}^{*}\cdot(\underset{\sim}{k}\times d\underset{\sim}{f})} \quad = \quad 0 \quad = \quad \overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\cdot d\underset{\sim}{f}} \qquad (6.156c)$$

$$\overline{d\underset{\sim}{f}^{*}\times(\underset{\sim}{k}\times d\underset{\sim}{f})} \quad = \quad \underset{\sim}{k}\,\frac{\psi}{2\pi k^2} \quad = \quad -\overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\times d\underset{\sim}{f}} \qquad (6.156d)$$

$$\overline{d\underset{\sim}{f}^{*}\times d\underset{\sim}{f}} \quad = \quad 0 \quad = \quad \overline{(\underset{\sim}{k}\times d\underset{\sim}{f}^{*})\times(\underset{\sim}{k}\times d\underset{\sim}{f})} \qquad (6.156e)$$

Substituting (6.156a-e) into equations (6.152)-(6.154), we obtain

$$\overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\cdot(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} = \left|\frac{\mathcal{J}}{\rho D}\right|^2 \frac{\psi}{2\pi}\left\{|\sigma|^2 k^2 + 4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^2\right\} \qquad (6.157)$$

$$\overline{\{\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}\}\cdot\{i\underset{\sim}{k}\times\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}\}} = \left|\frac{\mathcal{J}}{\rho D}\right|^2 \frac{\psi}{2\pi}\, 4k^2(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})\,\mathcal{I}m\,\sigma \qquad (6.158)$$

$$\overline{\{\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}\}\times\{\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\}} = \qquad (6.159)$$

$$= -\left|\frac{\mathcal{J}}{\rho D}\right|^2 \frac{\psi}{2\pi}\,\frac{(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})}{(\eta k^2+i\omega)}\,\underset{\sim}{k}\left\{4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})\,\mathcal{I}m\,\sigma\right\}$$

As may be seen from equations (6.147) and (6.148), $|\sigma|^2$ and $|D|^2$ are *even* functions of ω, while $\text{Im}\,\sigma$ is an *odd* function of ω. Thus, when the expressions (6.157)-(6.159) are integrated over the range $-\infty$ to $+\infty$ in ω, (6.157) will be the only term to give a nonzero result, if $\psi(k,\omega)$ is an *even* function of ω. Both the *helicity*, given by (6.150), and the *fluctuating e.m.f.*, given by (6.151), are therefore zero to first order when $\psi(k,\omega)$ is *even* in ω.

Moffatt (1972) has considered equations (6.157)-(6.159) in some detail, and has obtained nonzero helicity and fluctuating e.m.f. by assuming that

$$\psi(k,\omega) \equiv 0 \quad \text{when} \quad \omega(\underset{\sim}{k}\cdot\underset{\sim}{\Omega}) < 0 \tag{6.160}$$

He has then examined the asymptotic behaviour of the kinetic and magnetic energy of the system in the limit of large times. In this thesis, however, we shall retain the assumption that F' is *locally isotropic* - i.e.

$$\psi(k,\omega) = \psi(k,-\omega) \tag{6.161}$$

and examine the effects of *inhomogeneity* and of *nonzero mean flow*.

6.4.4 The second-order fluctuating equations and their solutions

From equations (6.136)-(6.139), and the assumption that terms involving $\bar{\underset{\sim}{u}}$ contribute only to second order, the second-order fluctuating dynamo equations are

$$(i\omega+\eta k^2)\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)} = 2i\eta\underset{\sim}{k}\cdot\underset{\sim}{\nabla}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} + i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{B}})\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)} + \bar{\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} - i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} - (\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})\cdot\underset{\sim}{\nabla}\bar{\underset{\sim}{B}} \tag{6.162}$$

$$(i\omega+\nu k^2)\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)} + 2\underset{\sim}{\Omega}\times\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)} = \tag{6.163}$$
$$= 2i\nu\underset{\sim}{k}\cdot\underset{\sim}{\nabla}\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} - \frac{1}{\rho}\underset{\sim}{\nabla}P\,dp - i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})\underset{\approx}{U}\cdot d\underset{\sim}{Z}_0 + \frac{1}{\rho\mu}\left\{\left[i\underset{\sim}{k}\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)} + \underset{\sim}{\nabla}\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\right]\times\bar{\underset{\sim}{B}} + (\underset{\sim}{\nabla}\times\bar{\underset{\sim}{B}})\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\right\}$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)} = -\underset{\sim}{\nabla}\cdot\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} \tag{6.164}$$

$$i\underset{\sim}{k}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)} = -\underset{\sim}{\nabla}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} - \{\underset{\sim}{\nabla}\rho/\rho\}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} \tag{6.165}$$

Solving these equations for $\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)}$ and $\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)}$, we obtain

$$\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)} = \frac{1}{i\omega+\eta k^2}\left\{i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{B}})\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)} + 2i\eta\underset{\sim}{k}\cdot\underset{\sim}{\nabla}\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} + \bar{\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} - i(\underset{\sim}{k}\cdot\bar{\underset{\sim}{u}})\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} - (\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})\cdot\nabla\bar{\underset{\sim}{B}}\right\} \tag{6.166}$$

$$\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)} = \frac{1}{D}\left\{2(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})d\underset{\sim}{H}_0 + \sigma(\underset{\sim}{k}\times d\underset{\sim}{H}_0) + i\sigma^2\underset{\sim}{k}\left[\underset{\sim}{\nabla}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} + (\underset{\sim}{\nabla}\rho/\rho)\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}\right]\right\} \tag{6.167}$$

where

$$d\underset{\sim}{H}_0 \equiv -2i\nu\,\underset{\sim}{k}\times(\underset{\sim}{k}\cdot\underset{\sim}{\nabla})\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} + \frac{1}{\rho}\underset{\sim}{k}\times\underset{\sim}{\nabla}P\,dp + \tag{6.168}$$

$$+\; i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})\underset{\sim}{k}\times\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} + 2i\underset{\sim}{\Omega}\{\underset{\sim}{\nabla}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} + (\underset{\sim}{\nabla}\rho/\rho)\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}\} +$$

$$\cdot\cdot\;\frac{1}{\rho\mu}\left\{\frac{i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})}{i\omega+\eta k^2}\right\}\underset{\sim}{k}\times\{2i\eta(\underset{\sim}{k}\cdot\underset{\sim}{\nabla})\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} + \overline{\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} +$$

$$- i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} - (\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{B}}\} +$$

$$-\frac{1}{\rho\mu}\underset{\sim}{k}\times\{(\underset{\sim}{\nabla}\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)})\times\overline{\underset{\sim}{B}} + (\underset{\sim}{\nabla}\times\overline{\underset{\sim}{B}})\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\}$$

The occurrence of nonzero second-order contributions to the helicity and the fluctuating e.m.f. is guaranteed by the nature of the second-order terms in (6.150) and (6.151). From equations (6.145) and (6.146),

$$[\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}] = [\text{even in }\omega] + i\,[\text{odd in }\omega] \tag{6.169}$$

$$[\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}] = [\text{odd in }\omega] + i\,[\text{even in }\omega] \tag{6.169'}$$

Similarly, from equation (6.168),

$$[d\underset{\sim}{H}_0] = [\text{odd in }\omega] + i\,[\text{even in }\omega] + [\tfrac{1}{\rho}\underset{\sim}{k}\times\underset{\sim}{\nabla}P\,dp]$$

so that

$$[\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(1)}] = [\text{odd in }\omega] + i\,[\text{even in }\omega] + [\underset{\sim}{a}_1 dp] \tag{6.170}$$

$$[\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)}] = [\text{even in }\omega] + i\,[\text{odd in }\omega] + [\underset{\sim}{a}_2 dp] \tag{6.170'}$$

If we assume that the fluctuating pressure and force distributions are uncorrelated - i.e. if

$$\overline{df_i^{*}dp} \equiv 0 \equiv \overline{dp^{*}df_i}\quad,\ \forall\, i \tag{6.171}$$

- then it follows from (6.169)-(6.170') that

$$\overline{[\underset{\approx}{u}\cdot d\underset{\sim}{Z}^{(0)}]^{*}[\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}]} = [\text{odd in } \omega] + i\,[\text{even in } \omega]$$

$$\overline{[\underset{\approx}{u}\cdot d\underset{\sim}{Z}^{(1)}]^{*}[\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}]} = [\text{even in } \omega] + i\,[\text{odd in } \omega]$$

$$\overline{[\underset{\approx}{u}\cdot d\underset{\sim}{Z}^{(0)}]^{*}[\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)}]} = [\text{even in } \omega] + i\,[\text{odd in } \omega] \qquad (6.172)$$

$$\overline{[\underset{\approx}{u}\cdot d\underset{\sim}{Z}^{(1)}]^{*}[\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)}]} = [\text{odd in } \omega] + i\,[\text{even in } \omega]$$

etc.

From (6.172), (6.150), and (6.151) we see that only *even-order* contributions (i.e. second-order, fourth-order, etc.) to the helicity and the fluctuating e.m.f. can be nonzero.

6.4.5 Second-order contributions to $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ in the geodynamo

It is clear from equations (6.166)-(6.168) that there are a large number of terms which contribute to $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ in the second-order approximation. We shall scale these terms in order to determine which of them are important in the geodynamo.

With the scaling

$$[k] \sim 1/\ell \quad , \quad [\omega] \sim 1/\tau \quad , \quad [\nabla] \sim 1/L \tag{6.173}$$

and the definitions

$$V_A \equiv \bar{B}/\sqrt{\rho\mu} \tag{6.174}$$

$$R_A \equiv V_A \ell/\eta \tag{6.175}$$

$$R_m \equiv \bar{u} L/\eta \tag{6.176}$$

$$q \equiv \ell^2/\eta\tau \tag{6.177}$$

equation (6.147) reduces to

$$[\sigma] \sim \frac{\eta}{\ell^2}\left\{\left[\frac{R_A}{(1+q^2)} + \frac{\nu}{\eta}\right] + iq\left[\frac{R_A}{(1+q^2)} + 1\right]\right\} \tag{6.178}$$

Similarly, equation (6.168) scales as

$$[d\underset{\sim}{H}_o] \sim i\frac{u'\eta}{\ell^2 L}\left\{\underset{①}{\left[\frac{R_A^2 R_m}{(1+iq)^2}\right]} + \underset{②}{[R_m]} + \underset{③}{[\Omega\ell^2/\eta]} + \underset{④}{\left[\frac{R_A^2}{(1+iq)}\right]} + \right.$$

$$\left. + \underset{⑤}{\left[\frac{R_A^2}{(1+iq)^2}\right]} + \underset{⑥}{[\nu/\eta]}\right\} + \left[\frac{1}{\rho}\underset{\sim}{k} \times \underset{\sim}{\nabla} P dp\right] \tag{6.179}$$

where

$$① = -\frac{1}{\rho\mu}\frac{\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}}}{(i\omega+\eta k^2)}\underset{\sim}{k}\times(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})\,\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} \tag{6.180a}$$

$$② = i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})\,\underset{\sim}{k}\times\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} \tag{6.180b}$$

$$③ = 2i\underset{\sim}{\Omega}\left\{\underset{\sim}{\nabla}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} + (\underset{\sim}{\nabla}\rho/\rho)\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}\right\} \tag{6.180c}$$

$$\begin{aligned}④ = -\frac{1}{\rho\mu}\Big\{&\frac{i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})}{i\omega+\eta k^2}\underset{\sim}{k}\times\left[\overline{\underset{\sim}{B}}\cdot\underset{\sim}{\nabla}\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} - (\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})\cdot\underset{\sim}{\nabla}\overline{\underset{\sim}{B}}\right]\\ &+\underset{\sim}{k}\times\left[(\underset{\sim}{\nabla}\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)})\times\overline{\underset{\sim}{B}} + (\underset{\sim}{\nabla}\times\overline{\underset{\sim}{B}})\times\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\right]\Big\}\end{aligned} \tag{6.180d}$$

$$⑤ = -\frac{1}{\rho\mu}\left\{\frac{i(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})}{i\omega+\eta k^2}\right\}\underset{\sim}{k}\times\left\{2i\eta(\underset{\sim}{k}\cdot\underset{\sim}{\nabla})\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}\right\} \tag{6.180e}$$

$$⑥ = -2i\nu\,\underset{\sim}{k}\times(\underset{\sim}{k}\cdot\underset{\sim}{\nabla})\,\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} \tag{6.180f}$$

In the fluid core of the Earth, we may take

$$\eta \sim 3\ \mathrm{m^2/sec} \tag{6.181a}$$

$$\nu \sim 6 \times 10^{-7}\ \mathrm{m^2/sec} \tag{6.181b}$$

$$L \sim 3 \times 10^{6}\ \mathrm{m} \tag{6.181c}$$

$$\bar{u} \sim 1 \times 10^{-4}\ \mathrm{m/sec} \tag{6.181d}$$

$$\Omega \sim 7 \times 10^{-5}\ \mathrm{rad/sec} \tag{6.181e}$$

$$R_m \sim 10^{2} \tag{6.181f}$$

$$V_A \sim 10^{-1}\ \mathrm{m/sec} \tag{6.181g}$$

making use of the estimates (6.15), (6.82), and (6.100), taking the value of $\bar{u}$ suggested by *Roberts and Soward*

(1972), and taking the estimate for the *Alfvén speed*, V_A , given by *Acheson and Hide (1973)*. From (6.175), (6.181a), and (6.181g) we see that

$$R_A \sim \ell/30 > 1 \quad \textit{if} \quad \ell > 30 \text{ m} \tag{6.182}$$

Also, from (6.181a,b) we see that

$$\frac{\nu}{\eta} << 1 \tag{6.183}$$

in the main body of the core if the *Gans (1972a)* estimate of ν is valid.

In considering equation (6.179), we may distinguish three possible sets of conditions.

$$\begin{array}{ll} \text{(A)} & \underline{\ell < 30\sqrt{1+q^2} \text{ m. and } q < 70} \\ \text{OR} & \underline{\ell < 2 \text{ km. and } q > 70} \end{array} \tag{6.184}$$

Under these conditions, the various terms in equation (6.179) stand in the relationship:

$$\begin{array}{l} [②] > [①] > [⑤] \\ [②] > [③] \\ [②] > [④] \end{array} \tag{6.184'}$$

The leading term in (6.179) is thus ② , and

$$d\underset{\sim}{H}_o^{(A)} \approx i(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})\underset{\sim}{k}\times\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(o)} + \frac{1}{\rho}\underset{\sim}{k}\times\underset{\sim}{\nabla}P dp \tag{6.184"}$$

is the appropriate approximation for $d\underset{\sim}{H}_o$.

(B) $\ell > 30\sqrt{1+q^2}$ m. and $q < 70$ (6.185)

Under these conditions,

$$[①] > [②]$$
$$[①] > [③]$$
$$[①] > [④] \qquad (6.185')$$
$$[①] > [⑤]$$

so that

$$d\underset{\sim}{H}_0^{(B)} \approx -\frac{1}{\rho\mu}\frac{(\underset{\sim}{k}\cdot\underset{\approx}{\overline{B}})}{(i\omega+\eta k^2)}\underset{\sim}{k}\times(\underset{\sim}{k}\cdot\underset{\approx}{\overline{U}})\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)} + \qquad (6.185'')$$
$$+ \frac{1}{\rho}\underset{\sim}{k}\times\underset{\sim}{\nabla}P\,dp$$

(C) $\ell > 2000$ m. and $q > 70$ (6.186)

Under these conditions

$$[③] > [①] > [⑤]$$
$$[③] > [②] \qquad (6.186')$$
$$[③] > [④]$$

so that

$$d\underset{\sim}{H}_0^{(C)} \approx 2i\underset{\sim}{\Omega}\left\{\underset{\sim}{\nabla}\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)} + (\underset{\sim}{\nabla}\rho/\rho)\cdot\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}\right\} \qquad (6.186'')$$
$$+ \frac{1}{\rho}\underset{\sim}{k}\times\underset{\sim}{\nabla}P\,dp$$

In all three cases, (A), (B), and (C), term ⑥ has been ignored. This approximation is certainly valid in the

geodynamo, since [⑥] < [②] even when the estimate obtained in *section 6.3.4* for the effective kinematic viscosity near the core-mantle interface is used.

We may also note that in equation (6.166)

$$[2i\eta \underset{\sim}{k} \cdot \underset{\sim}{\nabla} \underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(0)}] \sim \frac{2}{R_m} [i(\underset{\sim}{k} \cdot \overline{\underset{\sim}{u}}) \underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(0)}]$$

Since $R_m >> 1$ in the geodynamo, we may ignore the term on the left hand side. Equation (6.166) thus reduces to

$$\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(1)} \approx \frac{i(\underset{\sim}{k} \cdot \overline{\underset{\sim}{B}})}{(i\omega + \eta k^2)} \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(1)} + \frac{(\underset{\sim}{k} \cdot \overline{\underset{\sim}{u}})(\underset{\sim}{k} \cdot \overline{\underset{\sim}{B}})}{(i\omega + \eta k^2)^2} \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)} + \frac{1}{(i\omega + \eta k^2)} \left\{ (\overline{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla}) \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)} - (\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}) \cdot \underset{\sim}{\nabla} \overline{\underset{\sim}{B}} \right\} \tag{6.187}$$

Scaling the terms in (6.187), we obtain

$$[\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(1)}] \sim \frac{\overline{B} \ell}{\eta(1+iq)} \left\{ [\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(1)}] + \left[\frac{R_m}{(1+iq)} + \frac{\ell}{L} \right] [\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}] \right\} \tag{6.187'}$$

Examination of (6.187') leads to the conclusion that the terms in braces on the right hand side of (6.187) can be neglected, provided that

$$q < R_m (L/\ell) \tag{6.188}$$

Equation (6.187) may then be rewritten

$$\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(1)} \approx \frac{i(\underset{\sim}{k} \cdot \overline{\underset{\sim}{B}})}{(i\omega + \eta k^2)} \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(1)} + \frac{(\underset{\sim}{k} \cdot \overline{\underset{\sim}{u}})(\underset{\sim}{k} \cdot \overline{\underset{\sim}{B}})}{(i\omega + \eta k^2)^2} \underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)} \tag{6.189}$$

We may now evaluate the second-order terms in the integral expression for $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ in the geodynamo for each of the three cases (A), (B), and (C). From equations (6.155) and (6.146),

$$\overline{\{\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}\} \times \{\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}\}} = 2i \left|\frac{\mathfrak{F}}{\rho D}\right|^2 \frac{\psi}{2\pi} (\underset{\sim}{k} \cdot \underset{\sim}{\Omega}) \,\mathcal{I}m\, \sigma \,\underset{\sim}{k} \tag{6.190}$$

$$\overline{\{\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}\} \cdot \{\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}\}} = \left|\frac{\mathfrak{F}}{\rho D}\right|^2 \frac{\psi}{2\pi} \left\{|\sigma|^2 k^2 + 4(\underset{\sim}{k} \cdot \underset{\sim}{\Omega})^2\right\} \tag{6.191}$$

The second-order terms in the integrand of equation (6.151) are therefore given by

$$\mathcal{R}e\left\{ \overline{[\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(1)*}] \times [\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(0)}]} + \overline{[\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}] \times [\underset{\approx}{\beta} \cdot dY^{(1)}]} \right\} =$$

$$= \mathcal{R}e\left\{ \frac{i(\underset{\sim}{k} \cdot \underset{\sim}{\overline{B}})}{(i\omega + \eta k^2)} \left[\overline{(\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(1)*}) \times (\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)})} + \overline{(U^* \cdot dZ^{(0)*}) \times (U \cdot dZ^{(1)})} \right] + \right.$$

$$\left. + \frac{(\underset{\sim}{k} \cdot \underset{\sim}{\overline{u}})(\underset{\sim}{k} \cdot \underset{\sim}{\overline{B}})}{(i\omega + \eta k^2)^2} \overline{[\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}] \times [\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}]} \right\}$$

$$= -\frac{2\eta k^2 (\underset{\sim}{k} \cdot \underset{\sim}{\overline{B}})}{(\eta^2 k^4 + \omega^2)} \,\mathcal{I}m \left\{ \overline{[\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(1)*}] \times [\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)}]} \right\} +$$

$$+ \frac{4\eta k^2 \omega (\underset{\sim}{k} \cdot \underset{\sim}{\overline{u}})(\underset{\sim}{k} \cdot \underset{\sim}{\overline{B}})}{(\eta^2 k^4 + \omega^2)^2} \left|\frac{\mathfrak{F}}{\rho D}\right|^2 \frac{\psi}{2\pi} (\underset{\sim}{k} \cdot \underset{\sim}{\Omega}) \,\mathcal{I}m\, \sigma \,\underset{\sim}{k} \tag{6.192}$$

In deriving (6.192), we have made use of equations (6.145), (6.189), and (6.190).

When case (A) is considered, so that $d\underset{\sim}{H}_o$ is given by equation (6.184"),

$$\mathcal{I}m\left\{\overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(1)*})^{(A)}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})}\right\} =$$

$$= \mathcal{I}m\left\{\frac{1}{D^{*}}\overline{\left[2(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})d\underset{\sim}{H}_{o}^{(A)*}+\sigma^{*}(\underset{\sim}{k}\times d\underset{\sim}{H}_{o}^{(A)*})\right]\times\left[\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}\right]}\, + \right.$$

$$\left. -\frac{\sigma^{*2}}{D^{*}}\overline{\left[\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}+(\underset{\sim}{\nabla}\rho/\rho)\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}\right]\underset{\sim}{k}\times\left[\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}\right]}\right\}$$

$$\begin{aligned} = \mathcal{I}m\,\frac{1}{D^{*}}\Big\{ & -2i(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})(\underset{\sim}{k}\times\overline{\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}}) + \\ & -i\sigma^{*}(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})[\underset{\sim}{k}\times(\underset{\sim}{k}\times\overline{\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})]\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}}) + \\ & -i(\sigma^{*})^{2}\overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}+\frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\underset{\sim}{k}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})}\Big\} \end{aligned} \tag{6.193}$$

$$\begin{aligned} = \mathcal{I}m\,\frac{1}{D^{*}}\Big\{ & 2i(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})\underset{\sim}{k}\left|\frac{\mathcal{F}}{\rho D}\right|^{2}\frac{\psi}{2\pi}\left[|\sigma|^{2}k^{2}+4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^{2}\right] + \\ & -2\sigma^{*}\mathcal{I}m\sigma\,(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})k^{2}\left|\frac{\mathcal{F}}{\rho D}\right|^{2}\frac{\psi}{2\pi}\underset{\sim}{k} + \\ & -i(\sigma^{*})^{2}\overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}+\frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\underset{\sim}{k}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})}\Big\} \end{aligned}$$

$$\begin{aligned} = \left|\frac{\mathcal{F}}{\rho D^{2}}\right|^{2}\Big\{ & 2\,\mathcal{R}e D(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})\frac{\psi}{2\pi}\left[|\sigma|^{2}k^{2}+4\underset{\sim}{k}\cdot\underset{\sim}{\Omega}\right]\underset{\sim}{k} + \\ & +2\,\mathcal{R}e D(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})\frac{\psi}{2\pi}(\mathcal{I}m\sigma)^{2}k^{2}\underset{\sim}{k} + \\ & +2\,\mathcal{I}m D(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})\frac{\psi}{2\pi}\mathcal{I}m\sigma\,\mathcal{R}e\sigma\,k^{2}\underset{\sim}{k}\Big\} + \\ & -\mathcal{I}m\,i\frac{(\sigma^{*})^{2}}{D^{*}}\overline{\left\{\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}+\frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*}\right\}\underset{\sim}{k}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} \end{aligned} \tag{6.194}$$

From (6.193), (6.178), and (6.184), the terms on the right hand side of equation (6.194) scale as

$$\frac{u'^2 \sigma \bar{u}}{\ell^3 D} \left\{ \frac{\ell^2 \Omega}{\eta(\nu/\eta + iq)} + 1 + \frac{1}{R_m}(\nu/\eta + iq) \right\}$$

It is therefore reasonable to neglect the last group of terms in (6.194) - i.e. those under the "average" bar - whenever

$$q < R_m \sim 100 \tag{6.195}$$

(6.195) will be valid under the first set of conditions characterizing case (A) *(see equation 6.184)*.

Substituting equation (6.194) into (6.192), we see that in case (A)

<u>if $\ell < 30\sqrt{1+q^2}$ m. , $q < 70$</u>

$$\mathcal{R}e \left\{ (\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(1)*}) \times (\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(0)}) + (\underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*}) \times (\underset{\approx}{\beta} \cdot d\underset{\sim}{Y}^{(1)}) \right\} \approx$$

$$\approx \left| \frac{\mathcal{F}}{\rho D} \right|^2 \frac{\psi}{2\pi} (\underset{\sim}{k} \cdot \underset{\sim}{\Omega})(\underset{\sim}{k} \cdot \bar{\underset{\sim}{u}})(\underset{\sim}{k} \cdot \bar{\underset{\sim}{B}}) \frac{4\eta k^2}{\eta^2 k^4 + \omega^2} \underset{\sim}{k} \cdot$$

$$\cdot \left\{ -\mathcal{R}e D \left[\frac{|\sigma|^2 k^2 + 4(\underset{\sim}{\Omega} \cdot \underset{\sim}{k})^2}{|D|^2} + \frac{(\mathcal{I}m\,\sigma)^2 k^2}{|D|^2} \right] + \right. \tag{6.196}$$

$$\left. + \mathcal{I}m D\, \mathcal{I}m\,\sigma\, \mathcal{R}e\,\sigma \frac{k^2}{|D|^2} + \frac{2\omega\, \mathcal{I}m\,\sigma}{\eta^2 k^4 + \omega^2} \right\}$$

<u>if $\ell < 2\,km$, $q > 100$</u>

$$\approx \left| \frac{\mathcal{F}}{\rho D} \right|^2 \frac{\psi}{2\pi} (\underset{\sim}{k} \cdot \underset{\sim}{\Omega})(\underset{\sim}{k} \cdot \bar{\underset{\sim}{u}})(\underset{\sim}{k} \cdot \bar{\underset{\sim}{B}}) \frac{4\eta k^2 \omega}{(\eta^2 k^4 + \omega^2)^2} \mathcal{I}m\,\sigma\, \underset{\sim}{k} +$$

$$+ \frac{2\eta k^2 (\underset{\sim}{k} \cdot \bar{\underset{\sim}{B}})}{(\eta^2 k^4 + \omega^2)} \mathcal{R}e \left\{ \frac{\sigma^{*2}}{D^*} \overline{\left(\underset{\sim}{\nabla} \cdot \underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*} + \frac{1}{\rho} \underset{\sim}{\nabla} \rho \cdot \underset{\approx}{U}^* \cdot d\underset{\sim}{Z}^{(0)*} \right) \underset{\sim}{k} \times (\underset{\approx}{U} \cdot d\underset{\sim}{Z}^{(0)})} \right\}$$

(6.197)

Similarly, when case (B) is considered, so that $d\underset{\sim}{H}_o$ is given by equation (6.185"),

$$\mathcal{I}m\left\{ \overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(1)*})^{(B)} \times (\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} \right\} = \tag{6.198}$$

$$= \mathcal{I}m\left\{ \frac{2i}{\rho\mu D^{*}} \frac{(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\overline{u}})(\underset{\sim}{k}\cdot\underset{\sim}{\overline{B}})^{2}}{(\eta k^{2}-i\omega)^{2}} \overline{(\underset{\sim}{k}\times\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} + \right.$$

$$+ \frac{i\sigma^{*}}{\rho\mu D^{*}} \frac{(\underset{\sim}{k}\cdot\underset{\sim}{\overline{u}})(\underset{\sim}{k}\cdot\underset{\sim}{\overline{B}})^{2}}{(\eta k^{2}-i\omega)^{2}} \overline{[\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})]\times[\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)}]} +$$

$$\left. - \frac{i\sigma^{*2}}{D^{*}} \overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*} + \frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\underset{\sim}{k}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} \right\}$$

Scaling the terms on the right hand side of (6.198), we obtain

$$\frac{u'^{2}|\sigma|\overline{u}}{\ell^{3}D^{*}} \left\{ \frac{2\Omega\ell^{2}}{\eta(1+q)} + \frac{R_A^{2}}{(1+q^{2})} + \frac{(1+q)\,R_A^{2}}{(1+q^{2})\,Rm} \right\}$$

The second term is dominant under the conditions specified in (6.185). Substituting (6.198) into (6.192), we obtain

<u>if $\ell > 30\sqrt{1+q^{2}}$ m. , $q < 70$</u>

$$\mathcal{R}e\left\{ \overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(1)*})\times(\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)})} + \overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\times(\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)})} \right\} \approx$$

$$\approx \left|\frac{\mathcal{F}}{\rho D}\right|^{2} (\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\overline{u}})(\underset{\sim}{k}\cdot\underset{\sim}{\overline{B}}) \frac{4\eta k^{2}}{(\eta^{2}k^{4}+\omega^{2})^{2}} \frac{\psi}{2\pi} \mathcal{I}m\sigma \; \underset{\sim}{k} \; \cdot \tag{6.199}$$

$$\cdot\left\{ \omega - \frac{2}{|D|^{2}} \omega\eta k^{4} \frac{(\underset{\sim}{k}\cdot\underset{\sim}{\overline{B}})^{2}}{\rho\mu(\eta^{2}k^{4}+\omega^{2})} (\mathcal{R}e\sigma\,\mathcal{R}eD + \mathcal{I}m\sigma\,\mathcal{I}mD) + \right.$$

$$\left. - \frac{1}{|D|^{2}} k^{2}(\eta^{2}k^{4}-\omega^{2}) \frac{(\underset{\sim}{k}\cdot\underset{\sim}{\overline{B}})^{2}}{\rho\mu(\eta^{2}k^{4}+\omega^{2})} (\mathcal{R}e\sigma\,\mathcal{I}mD - \mathcal{I}m\sigma\,\mathcal{R}eD) \right\}$$

Finally, when case (C) is considered, so that $d\underset{\sim}{H}_o$ is given by equation (6.186"),

$$\mathcal{I}m\left\{ \overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(1)*})^{(c)} \times (\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} \right\} = \tag{6.200}$$

$$= \mathcal{I}m\left\{ -\frac{2(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})}{D^{*}}\, 2i\, \overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*} + \frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\underset{\sim}{\Omega}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} + \right.$$

$$- 2i\frac{\sigma^{*}}{D^{*}}(\underset{\sim}{k}\times\underset{\sim}{\Omega})\times\overline{(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*} + \frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})} +$$

$$\left. - i\frac{(\sigma^{*})^2}{D^{*}}\overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*} + \frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\underset{\sim}{k}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})} \right\}$$

Scaling the terms on the right hand side of (6.200), we obtain

$$\frac{u'^2\Omega^2}{\ell L|D|}\left\{ 1 + \frac{|\sigma|}{\Omega} + \frac{|\sigma|^2}{\Omega^2} \right\}$$

$$\sim \frac{u'^2\Omega^2}{\ell L|D|}\left\{ 1 + \frac{\eta R_A^2}{\ell^2\Omega q} + \frac{\eta^2 R_A^4}{\ell^4\Omega^2 q^2} \right\} \quad \text{if } V_A > \ell/\tau$$

$$\sim \frac{u'^2\Omega^2}{\ell L|D|}\left\{ 1 + (\tau\Omega)^{-1} + (\tau\Omega)^{-2} \right\} \quad \text{if } V_A < \ell/\tau$$

Under the conditions specified in (6.186), (6.182), and (6.181), the first term on the right hand side of (6.200) is dominant when $V_A > \ell/\tau$. When $V_A < \ell/\tau$, on the other hand, the first term on the right hand side of (6.200) will dominate only if $\tau > 1$ day . Substituting (6.200) into (6.192), and assuming that the first term in (6.200) is dominant, we obtain

if $\ell > 2\text{km}$, $q > 70$ (and $\tau > 1$ day, if $V_A < \ell/\tau$)

$$\mathcal{R}e\left\{\overline{(\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(1)*})\times(\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(0)}) + (\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)})\times(\underset{\approx}{\beta}\cdot d\underset{\sim}{Y}^{(1)})}\right\} \approx$$

$$\approx \frac{4\eta k^2}{(\eta^2 k^4+\omega^2)}(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\underset{\sim}{\bar{B}}) \cdot \tag{6.201}$$

$$\cdot\left\{ 2\,\mathcal{R}e\left[\frac{1}{D^*}\overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*} + \frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^{*}\cdot d\underset{\sim}{Z}^{(0)*})\,\underset{\sim}{\Omega}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})}\right] + \right.$$

$$\left. + \left|\frac{\mathcal{J}}{\rho D}\right|^2 \frac{\Psi}{2\pi}(\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})\frac{\omega}{\eta^2 k^4+\omega^2}\,\mathcal{I}m\sigma\;\underset{\sim}{k}\right\}$$

In equations (6.196), (6.197), (6.199), and (6.201), we have, from (6.147) and (6.148),

$$\mathcal{I}m\,\sigma = \omega\left\{1 - \frac{(\underset{\sim}{k}\cdot\underset{\sim}{\bar{B}})^2}{\rho\mu(\eta^2 k^4+\omega^2)}\right\} \tag{6.202}$$

$$\mathcal{R}e\,\sigma = \eta k^2\left\{\frac{\nu}{\eta} + \frac{(\underset{\sim}{k}\cdot\underset{\sim}{\bar{B}})^2}{\rho\mu(\eta^2 k^4+\omega^2)}\right\} \tag{6.202'}$$

$$\mathcal{I}m\,D = 2k^2\,\mathcal{I}m\sigma\,\mathcal{R}e\,\sigma \tag{6.203}$$

$$\mathcal{R}e\,D = k^2\left\{(\mathcal{R}e\sigma)^2 - (\mathcal{I}m\sigma)^2\right\} + 4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^2 \tag{6.203'}$$

We may now write explicit expressions for the fluctuating e.m.f. $\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'}$ in the various cases considered above. Substituting equations (6.196), (6.197), (6.199) and (6.201) into equation (6.151), we obtain

(A1) $\ell < 30\sqrt{1+q^2}$ m , $q < 70$

$$\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'} \approx \iint_{\underset{\sim}{k}\omega} d\underset{\sim}{k}\,d\omega \left|\frac{\mathcal{J}}{\rho D}\right|^2 \frac{\psi}{2\pi}\,\frac{4\eta k^2}{(\eta^2k^4+\omega^2)}\,(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})\,\underset{\sim}{k}\,\cdot$$

$$\cdot\left\{\frac{k^2}{|D|^2}\,\mathcal{I}m D\,\mathcal{I}m\sigma\,\mathcal{R}e\sigma - \frac{1}{|D|^2}\mathcal{R}e D\left[|\sigma|^2k^2+(\mathcal{I}m\sigma)^2k^2+4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^2\right] + \right.$$

$$\left. + \frac{2\omega\,\mathcal{I}m\sigma}{(\eta^2k^4+\omega^2)}\right\} \tag{6.204}$$

(A2) $\ell < 2$ km , $q > 100$

$$\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'} \approx \iint_{\underset{\sim}{k}\omega} d\underset{\sim}{k}\,d\omega \left|\frac{\mathcal{J}}{\rho D}\right|^2 \frac{\psi}{2\pi}\,\frac{4\eta k^2\omega}{(\eta^2k^4+\omega^2)^2}\,(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})\,\mathcal{I}m\sigma\,\underset{\sim}{k} +$$

$$+\iint_{\underset{\sim}{k}\omega} d\underset{\sim}{k}\,d\omega\,\frac{2\eta k^2}{(\eta^2k^4+\omega^2)}(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})\;\cdot \tag{6.205}$$

$$\cdot\,\mathcal{R}e\left\{\frac{\sigma^{*2}}{D^*}\,\overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^{(0)*}+\frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^{(0)*})\,\underset{\sim}{k}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(0)})}\right\}$$

(B) $\ell > 30\sqrt{1+q^2}$ m , $q < 70$

$$\overline{\underset{\sim}{u}'\times\underset{\sim}{B}'} \approx \iint_{\underset{\sim}{k}\omega} d\underset{\sim}{k}\,d\omega \left|\frac{\mathcal{J}}{\rho D}\right|^2 \frac{\psi}{2\pi}\,\frac{4\eta k^2}{(\eta^2k^4+\omega^2)^2}(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}})(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})\,\mathcal{I}m\sigma\,\underset{\sim}{k}\,\cdot$$

$$\cdot\left\{\omega - \frac{k^2(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})^2}{\rho\mu(\eta^2k^4+\omega^2)}\left[2\eta\omega k^2(\mathcal{R}e\sigma\,\mathcal{R}e D+\mathcal{I}m\sigma\,\mathcal{I}m D) + \right.\right.$$

$$\left.\left. + (\eta^2k^4-\omega^2)(\mathcal{R}e\sigma\,\mathcal{I}m D - \mathcal{I}m\sigma\,\mathcal{R}e D)\right]\right\}$$

(6.206)

(c) $\ell > 2$ km., $q > 70$ (and $\tau > 1$ day, if $V_A < \ell/\tau$)

$$\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'} \approx \iint_{\underset{\sim}{k}\omega} d\underset{\sim}{k}\, d\omega \, \frac{4\eta k^2}{(\eta^2 k^4 + \omega^2)} \, (\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}}) \cdot$$

$$\cdot \left\{ 2\,\mathrm{Re}\left[\frac{1}{D^*} \overline{(\underset{\sim}{\nabla}\cdot\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^{(\omega)*} + \frac{1}{\rho}\underset{\sim}{\nabla}\rho\cdot\underset{\approx}{U}^*\cdot d\underset{\sim}{Z}^{(\omega)*})\underset{\sim}{\Omega}\times(\underset{\approx}{U}\cdot d\underset{\sim}{Z}^{(\omega)})} \right] + \right.$$

$$\left. + \left|\frac{\mathcal{F}}{\rho D}\right|^2 \frac{\psi}{2\pi} (\underset{\sim}{k}\cdot\overline{\underset{\sim}{u}}) \frac{\omega}{(\eta^2 k^4 + \omega^2)} \,\mathrm{Im}\,\sigma\, \underset{\sim}{k} \right\} \qquad (6.207)$$

6.4.6 The effects of locally isotropic turbulence in the geodynamo

The expressions (6.204)-(6.207), along with equations (6.202)-(6.203'), allow us to comment on the possible effects of a turbulent force distribution in the Earth's fluid core. The first question which must be answered is: *what value of* q *is appropriate to the evaluation of the expressions for* $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$? In his study of the first-order terms, *Moffatt (1972)* has suggested that the largest contributions to the integration over ω will come from the "natural" frequencies of the undamped system - i.e. from the frequencies at which

$$\lim_{\nu,\eta \to 0} |D|^2 \equiv |D_o|^2 = 0 \tag{6.208}$$

From equations (6.202) and (6.203) it may be seen that these "natural" frequencies are given by the roots of

$$D_o = 4(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^2 - k^2\left\{\omega - \frac{(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})^2}{\rho\mu\omega}\right\}^2 = 0 \tag{6.209}$$

Solving equation (6.209), we obtain the expression

$$\omega_n = \pm\frac{(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})}{k} \pm \sqrt{\frac{(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})^2}{k^2} + \frac{(\underset{\sim}{k}\cdot\overline{\underset{\sim}{B}})^2}{\rho\mu}} \tag{6.210}$$

for the "natural" frequencies ω_n .

To a very crude approximation, it follows from equation (6.210) that

$$|\omega_n| \sim \sqrt{\Omega^2 + V_A^2/\ell^2} \tag{6.211}$$

The value of q implied by the "natural" frequencies is therefore

$$q_n \equiv \ell^2|\omega_n|/\eta \sim \sqrt{(\Omega\ell^2/\eta)^2 + R_A^2} \tag{6.212}$$

Checking equation (6.212) against the requirements specified in each of (6.204)-(6.207), we see that the "natural" frequencies ω_n are likely to be consistent with cases (A1) and (C), but not with cases (A2) or (B). Assuming that the crude approximation (6.211) is valid in some sense as an "average" over the range of integration in $\underset{\sim}{k}$, we may restrict consideration to equations (6.204) and (6.207) when the "natural" frequencies are dominant.

It must, however, be pointed out that the "natural" frequencies will not necessarily give the largest contributions to the integration over ω in equations (6.204)-(6.207). If the *spectrum function* $\psi(k,\omega)$ of the force distribution vanishes in the neighbourhood of the "natural" frequencies, or if the spectrum function is sharply peaked at a "driving" frequency ω_o , contributions from the "natural" frequencies may well be negligible. We may distinguish two possible sets of conditions:

a. $\psi(k,\omega)$ is a "broad band" spectrum function, with at least one of the "natural" frequencies ω_n included in the band in which ψ is nonzero.

b. $\psi(k,\omega)$ is sharply peaked at one or more "driving" frequencies.

When (a) is valid, contributions from the "natural" frequencies will dominate the integrals in (6.204) and (6.207), and the approximate method of integration suggested by *Moffatt (1972)* will be applicable. Because of the condition (6.212), it will not be necessary to consider equations (6.205) and (6.206). It should be noted that in the remaining two equations it will be necessary to include contributions from all four "natural" frequencies. *Moffatt (1972)* obtains contributions from only two of these frequencies because of his assumption that $\psi(k,\omega)$ satisfies the condition (6.160).

On the other hand, when (b) is valid contributions from the "driving" frequencies will dominate the integrals in (6.204)-(6.207), and it will be appropriate to replace $\psi(k,\omega)$ with a sum of terms of the form $\psi(k)\delta(\omega-\omega_o)$. As there are no restrictions on q in this case, it will be necessary to consider all four sets of conditions (6.204)-(6.207).

Because of the labour involved, we shall not carry the evaluation of $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ any further in this thesis. However, the principal features of the various contributions are already apparent from the equations in their present form. We shall first point out several general properties, shared by all four equations, (6.204)-(6.207).

[1] In each case, integration with respect to ω over the range $-\infty$ to $+\infty$ can give a nonzero result, since all the terms in the integrands are *even* functions of ω .

[2] In each case, the integral vanishes if $\eta = 0$. Thus *dissipation* is essential to the production of nonzero $\overline{\mathbf{u}' \times \mathbf{B}'}$ *(see section 1.4.2)*. This property is shared by the expressions obtained by *Moffatt (1972)*.

[3] In each case, the expression for $\overline{\mathbf{u}' \times \mathbf{B}'}$ gives an α-*effect* at low values of the mean field, $\overline{\mathbf{B}}$.

It should be noted that the integrands in equations (6.204)-(6.207) are nearly all *rotation-dependent*. The only exception is the second term in equation (6.205). When the necessary "averaging" in this term is carried out, we obtain

$$\overline{\left(\nabla\cdot \mathsf{U}^{*}\cdot d\mathbf{Z}^{(0)*} + \frac{1}{\rho}\nabla\rho\cdot \mathsf{U}^{*}\cdot d\mathbf{Z}^{(0)*}\right)\mathbf{k}\times\left(\mathsf{U}\cdot d\mathbf{Z}^{(0)}\right)} = \qquad (6.123)$$

$$= \frac{\psi}{4\pi k^{2}}\left\{\frac{\mathfrak{Z}\sigma}{\rho^{2}D}\,k^{4}\left[\mathbf{k}\times\nabla\left(\frac{\mathfrak{Z}\sigma}{D}\right)^{*}\right] + \frac{4\mathfrak{Z}}{\rho^{2}D}k^{2}(\mathbf{k}\cdot\boldsymbol{\Omega})^{2}\left[\mathbf{k}\times\nabla\left(\frac{\mathfrak{Z}}{D}\right)^{*}\right] + \right.$$

$$\left. + \frac{2\mathfrak{Z}}{\rho^{2}D}k^{2}(\mathbf{k}\cdot\boldsymbol{\Omega})\left[2i\,\mathfrak{Im}\,\sigma\,\nabla\left(\frac{\mathfrak{Z}}{D}\right)^{*} - \left(\frac{\mathfrak{Z}}{D}\right)^{*}\nabla\sigma^{*}\right]\cdot\left(\mathbf{k}\mathbf{k} - k^{2}\mathsf{I}\right)\right\}$$

making use of equations (6.146), (6.155), and (6.156). Thus, from equation (6.205), the contribution to $\overline{\mathbf{u}' \times \mathbf{B}'}$ which does not vanish with $\boldsymbol{\Omega}$ is

$$\{\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}\}_{\underset{\sim}{\Omega}=0} = \qquad (6.214)$$

$$= \iint_{\underset{\sim}{k}\,\omega} d\underset{\sim}{k}\, d\omega \, \frac{2\eta k^4}{(\eta^2 k^4 + \omega^2)} \frac{\Psi}{2\pi} (\underset{\sim}{k} \cdot \overline{\underset{\sim}{B}}) \, \mathcal{R}e \left\{ \frac{3|\sigma|^2 \sigma^*}{\rho^2 |D|^2} \underset{\sim}{k} \times \underset{\sim}{\nabla} \left(\frac{3\sigma}{D} \right)^* \right\}$$

subject to the conditions $\ell < 2$ km , $q > 100$. This term may be regarded as an example of the introduction of a fluctuating e.m.f. (and of *helicity*) through large-scale variation of the turbulent force intensity *(see the discussion in section 4.3.7)*.

The various terms in the integrands in equations (6.204)-(6.207) nearly all depend on the presence of a *mean flow* $\overline{\underset{\sim}{u}}$. The only exceptions are the second term in equation (6.205), discussed in the last paragraph, and the first term in equation (6.207). It may be shown *(see Table 17 below)* that the second term in (6.205) is dominant when $\overline{B}$ is large, while the first term in (6.207) is not. Since $\overline{B}$ may be considered large in the geodynamo, the importance of *mean flow* in turbulent dynamo action in the Earth appears to depend critically on the values of ℓ and τ appropriate to turbulence in the core.

The integrands in (6.204)-(6.207) show a variety of dependences on $\overline{\underset{\sim}{B}}$. These dependences are summarized in *Table 17*, overleaf. It can be seen from the table that most of the terms lead to a simple α-*effect* at low values of $\overline{B}$. However, the second and third terms in equation (6.206) have

a $\bar{B}^3$ dependence at low values of $\bar{B}$. At high values of $\bar{B}$ all terms are small, but the second term in equation (6.205), discussed in the last two paragraphs, falls off more slowly with increasing $\bar{B}$ than do the other terms.

TABLE 17

Dependence of integrands on $\bar{B}$ in expressions for $\underset{\sim}{u}' \times \underset{\sim}{B}'$

Equation	Term	Dependence on $\bar{B}$	
		Low $\bar{B}$	High $\bar{B}$
(6.204)	1	$\bar{B}$	$\bar{B}^{-7}$
	2	$\bar{B}$	$\bar{B}^{-7}$
	3	$\bar{B}$	$\bar{B}^{-5}$
(6.205)	1	$\bar{B}$	$\bar{B}^{-5}$
	2	$\bar{B}$	$\bar{B}^{-3}$
(6.206)	1	$\bar{B}$	$\bar{B}^{-5}$
	2	$\bar{B}^3$	$\bar{B}^{-5}$
	3	$\bar{B}^3$	$\bar{B}^{-5}$
(6.207)	1	$\bar{B}$	$\bar{B}^{-7}$
	2	$\bar{B}$	$\bar{B}^{-5}$

(It should be noted that in the last few paragraphs, and in *Table 17*, $\bar{B} \equiv |\underset{\sim}{\bar{B}}|$.)

We may now summarize the effects to be expected from a *locally isotropic* turbulent force distribution in the Earth's fluid core. The dominant effect of turbulent forces outside the boundary layer is determined in large part by the characteristic length scale ℓ of the turbulence, and by the ratio of the diffusion time on this length scale to the *effective* time scale of the turbulence. This *time ratio* is denoted by q .

If both ℓ and q are relatively "small" *(see equation 6.204)*, an α-*effect* appears at low values of $\bar{B}$ (the magnitude of the mean flux density). This effect depends for its existence on the presence of both *rotation* and a *mean flow*. At large values of $\bar{B}$, $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ goes to zero as $\bar{B}^{-5}$.

If both ℓ and q are relatively "large" *(see equation 6.207)*, a rotation-dependent α-effect again appears at low values of $\bar{B}$. Only part of this effect depends on the presence of a mean flow; the remainder depends on the presence of gradients of the turbulent force intensity. At large values of $\bar{B}$, $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ again goes to zero. However, the part of the effect which is *mean flow-dependent* disappears more gradually than the remainder, going to zero as $\bar{B}^{-5}$ rather than as $\bar{B}^{-7}$.

If ℓ is "small" while q is "large" *(see equation 6.205)*, an α-effect again appears at low values of $\bar{B}$.

One part of this effect depends for its existence on the presence of both rotation and a mean flow. A second part depends on the presence of both rotation and large-scale variations of the turbulent force intensity. The remainder of the effect depends *only* on the presence of large-scale variations of the turbulent force intensity. At large values of $\bar{B}$, all three parts of the effect disappear. However, the second and third parts fall off more slowly with increasing $\bar{B}$ than does the first ($\bar{B}^{-3}$ compared with $\bar{B}^{-5}$). In general, $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ is unlikely to have the behaviour described in this paragraph if the spectrum of the turbulent force distribution has a fairly uniform amplitude over a broad band of frequencies ω .

If ℓ is relatively "large" and q is relatively "small" *(see equation 6.206)*, an α-effect appears only at very small values of $\bar{B}$. At somewhat larger values of $\bar{B}$, $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ varies as $\bar{B}^3$. Both the α-effect and the $\bar{B}^3$ effect depend for their existence on the presence of rotation and mean flow. All terms contributing to $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ go to zero as $\bar{B}^{-5}$ at large values of $\bar{B}$. In general, $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ is unlikely to have the behaviour described in this paragraph if the spectrum of the turbulent force distribution is "broad-band" in ω .

6.5 Summary of Chapter 6

This chapter is concerned with the *hydromagnetic dynamo problem* as it applies to the Earth's fluid core. The *magnetogeostrophic* approximation is considered, and a general expression for the fluid velocity in the core is obtained, giving the velocity as a function of the distribution of body forces, the magnetic field, and the boundary conditions.

The equation obtained in *Chapter 5* for *boundary-layer control* of the external magnetic field is investigated in terms of the distribution of body forces at the core-mantle interface. It is shown that *radial forces* at the core-mantle interface cannot account for the observed temporal variations of the geomagnetic dipole moment. *Azimuthal forces*, on the other hand, can account for these variations. The azimuthal force term $-\rho(\dot{\underset{\sim}{\Omega}} \times \underset{\sim}{r})$ is shown to explain the *dipole wobble* reported by *Kawai and Hirooka (1967)* if the kinematic viscosity at the core-mantle interface is of the order $\nu \sim 1\text{-}2\ m^2/\text{sec}$. This term also leads to non-periodic variations in the axial dipole moment on time scales similar to those which characterize geomagnetic *polarity transitions*.

The effects of a turbulent distribution of body forces in the Earth's core are also considered. Turbulent forces outside the boundary layer produce a variety of effects. If the turbulent force distribution is *locally*

isotropic, the dominant effects are determined in large part by the characteristic length scale of the turbulence and the ratio of the diffusion time on this length scale to the *effective* time scale of the turbulence. In most cases a rotation-dependent α-effect is dominant when the mean magnetic flux density, $\bar{B}$, is small. This effect may also depend on the presence of a *mean flow*, or the presence of *large-scale variation* of the turbulent force intensity. At large values of $\bar{B}$, the fluctuating e.m.f. $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ goes to zero. The various types of behaviour which may occur are summarized at the end of *section 6.4.6*.

7. FINAL SUMMARY

The principal aims of this thesis have been to summarize present-day knowledge of astrophysical magnetic fields, and to discuss the possibility of their maintenance by dynamo action, with particular reference to the effects of turbulent distributions of force and of velocity.

Chapter 1 is devoted to an overall review of the dynamo problem, from both the observational and the theoretical points of view. *Schuster's hypothesis* concerning the magnetic fields of massive rotating bodies is also discussed. It is pointed out that this hypothesis has no experimental justification, and that it leads to incorrect predictions in certain cases. Extreme caution must therefore be employed when the hypothesis is used to predict the surface magnetic fields of bodies for which no observational data are available.

In *Chapter 2* a review of *mean field electrodynamics* is presented, and the mean field dispersion relation for "wave" mean fields is cast in a novel determinantal form. a new terminology is proposed for several types of homogeneous, stationary turbulence with particular invariance properties. It is hoped that this terminology will reduce the confusion which has sprung up in recent years in connection with the terms *isotropic* and *mirror-symmetric*.

Chapter 3 is devoted to the effects of *PT-invariant turbulence* (i.e. stationary, homogeneous turbulence whose average properties are invariant under space-time inversion) on magnetic fields which vary on scales larger than the turbulence correlation length and time. It is proved that the claim of *Lerche and Low (1971)* that isotropic turbulence in an incompressible fluid can support dynamo action is unfounded. Indeed, no PT-invariant turbulence can support dynamo action in an incompressible fluid, in the *first order smoothing* approximation.

The decay of "wave" mean fields in the presence of PT-invariant turbulence is also studied in *Chapter 3*. It is found that a number of conditions must be satisfied by the turbulence and the mean field if spatially periodic mean fields are to exist. The case of *isotropic Gaussian turbulence* is considered in detail, and several restrictions on the parameters of the turbulence and the mean field are derived. These restrictions can be interpreted as restrictions on the usefulness of the *Rädler expansion (Rädler, 1968; Krause and Rädler, 1971)* as a representation of the fluctuating e.m.f. $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$. It is also shown that the *Rädler expansion* is not useful when the mean field oscillates with time. Several restrictions on the existence of spatially periodic, oscillatory, decaying mean fields are derived, and the behaviour of fields satisfying these restrictions is studied numerically. It is suggested that

the numerical techniques used here may well provide the most convenient method for investigating dynamo action generated by *non-PT-invariant turbulence*. The possibility of a dynamo with *sporadic helicity* is also discussed.

Chapter 4 is concerned with nonstationary, inhomogeneous turbulence and its treatment within the framework of mean field electrodynamics. A successive approximation technique is developed, and is applied to the kinematic dynamo problem in this chapter. Later, in *Chapter 6*, the technique is used in an investigation of the hydromagnetic dynamo problem. The possibility of introducing *helicity* through large-scale variations of the turbulent velocity distribution is also discussed in *Chapter 4*. In *Chapter 6*, an example is provided in which helicity is produced by means of large-scale variations in the *turbulent body force distribution*.

In *Chapter 5* the problem of *time variations* of astrophysical magnetic fields is studied, and the suggestion is made that variations of this sort may well be subject to *boundary-layer control* in the geodynamo. This possibility first arises in connection with the "$\alpha^2(r)$" *dynamo in a spherical shell*, which is studied in detail. A more general study is then carried out, making use of a novel representation of the magnetic multipole moments of a spherical current distribution in terms of the integral moments of the internal magnetic flux density. It is shown

that for the geodynamo, *boundary-layer control* is only likely for field variations on time scales less than 10^4 years. *Dipole wobble* and large-scale variations of the axial magnetic dipole moment are shown to arise from certain distributions of velocity near the outer boundary of the Earth's fluid core.

In *Chapter 6* the idea of *boundary-layer control* of the external magnetic field of a spherical current distribution is studied in connection with the *hydromagnetic dynamo problem*. It is shown that *radial forces* at the core-mantle interface in the geodynamo cannot account for the observed temporal variations of the geomagnetic dipole moment. However, it is shown that the slow, systematic decrease of the Earth's speed of rotation can explain the observed *wobble* of the geomagnetic dipole axis if the kinematic viscosity at the core-mantle interface is of the order $\nu \sim 1\text{-}2\ m^2/sec$. Non-periodic variations in the axial dipole moment on time scales similar to those which characterize *geomagnetic polarity transitions* can also arise from the slow decrease of the Earth's speed of rotation.

The effects of a turbulent distribution of body forces in the Earth's fluid core are also considered in *Chapter 6*. Turbulent forces deep in the core can produce a variety of effects. The dominant effects are determined, in large part, by the characteristic length scale of the

turbulence and the ratio of the diffusion time on this length scale to the *effective* time scale of the turbulence. In most cases, a rotation-dependent α-effect is dominant at low values of the mean magnetic flux density, the effect being controlled either by the *mean flow* in the core or by gradients of the turbulent force intensity. However, under certain conditions other effects can be dominant. The relevance of these conditions to the Earth's core is discussed. At large values of the mean magnetic flux density, the fluctuating e.m.f. $\overline{\underset{\sim}{u}' \times \underset{\sim}{B}'}$ goes to zero. A detailed description of the types of behaviour which may occur is given at the end of *section 6.4.6*.

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APPENDIX 1

UNITS AND CONVERSION FACTORS

A.1.1 SI units

In this thesis, SI units are used as a general rule. For convenience, a summary of units commonly encountered is given in *Tables 18 and 19*. These tables include conversion factors relating SI units to unrationalized c.g.s. electromagnetic units. More complete summaries are to be found in many standard texts. *(See, for example, Stratton, 1941, pp. 601-603; Allen, 1963, pp. 21-29; Land, 1972, pp. 3-4.)*

It should be noted that in *Tables 1-6*, magnetic flux densities are quoted in *gauss* to facilitate comparison with values given in the literature on astrophysical magnetic fields. As may be seen from *Table 19*, the conversion factor which must be used to convert these flux densities to SI units is

$$1 \text{ gauss (G)} = 10^{-4} \text{ weber/m}^2 \text{ (Wb/m}^2\text{)} \qquad \text{(A1.1)}$$

$$= 10^{-4} \text{ tesla (T)} \qquad \text{(A1.1')}$$

A.1.2 Magnetic dipole moments

In the literature on astrophysical magnetic fields, magnetic dipole moments are frequently given in *gauss-cm*^{3} *(see, for example, Warwick, 1971; Sharp, Russell and*

Coleman, 1973). This unit has the dimensions $ML^3T^{-1}Q^{-1}$, in contrast to the dimensions $L^2T^{-1}Q$ quoted for magnetic moment in *Table 19*. The discrepancy lies in the definition of "magnetic dipole moment". In c.g.s. electromagnetic units, the "dipole moment" measured in *gauss-cm*^{3} has exactly the same magnitude as the "dipole moment" measured in *oersted-cm*^{3} (or *erg/gauss*), provided that $\mu = 1$. Since this condition is satisfied in most astrophysical situations, we may treat astrophysical dipole moments given in *gauss-cm*^{3} on the same footing as those given in *erg/gauss* (*oersted-cm*^{3}). The conversion factor to be used is

$$1 \text{ erg/gauss} = 10^{-3} \text{ ampere-m}^2 \ (\text{A}\cdot\text{m}^2) \qquad \text{(A1.2)}$$

as indicated in *Table 19*.

[It should be noted that several detailed discussions of the problem of units in electromagnetic theory have appeared in the last few years. A particularly useful account is given by *F. Primdahl*, in *Analysis of units in electromagnetism*, Publications of the Earth Physics Branch, Department of Energy, Mines and Resources, Ottawa, Canada, vol. 42, no. 1, 1971.]

TABLE 18 - MECHANICAL UNITS

Quantity	Symbol	Dimensions	SI unit	Conversion to cgs unit
Length		L	metre (m)	= 10^2 centimetre (cm)
Mass		M	kilogram (kg)	= 10^3 gram (g)
Time		T	second (sec, s)	= 1 second
Speed	u	LT^{-1}	m/sec	= 10^2 cm/sec
Density	ρ	ML^{-3}	kg/m^3	= 10^{-3} g/cm^3
Force		MLT^{-2}	newton (nt)	= 10^5 dyne
Pressure	p	$ML^{-1}T^{-2}$	nt/m^2	= 10 dyne/cm^2
Force density	F	$ML^{-2}T^{-2}$	nt/m^3	= 10^{-1} dyne/cm^3
Energy		ML^2T^{-2}	joule	= 10^7 erg
Angular frequency	ω,Ω	T^{-1}	radian/sec	= 1 radian/sec
Angular momentum	J	ML^2T^{-1}	joule·sec	= 10^7 erg·sec
Moment of inertia	I	ML^2	kg·m^2	= 10^7 g·cm^2
Bulk viscosity	$\rho\nu$	$ML^{-1}T^{-1}$	nt·sec/m^2	= 10 poise
Kinematic viscosity	ν	L^2T^{-1}	m^2/sec	= 10^4 stokes

TABLE 19 - ELECTROMAGNETIC UNITS

Quantity	Symbol	Dimensions	SI unit	Conversion to unrationalized cgs e.m.u.
Charge		Q	coulomb	$= 10^{-1}$ e.m.u.
Charge density	θ	$L^{-3}Q$	coulomb/m^3	$= 10^{-7}$ e.m.u.
Current		$T^{-1}Q$	ampere (A)	$= 10^{-1}$ e.m.u.
Current density	j	$L^{-2}T^{-1}Q$	A/m^2	$= 10^{-5}$ e.m.u.
Conductivity	σ	$M^{-1}L^{-3}TQ^2$	mho/m	$= 10^{-11}$ e.m.u.
Permeability	μ	MLQ^{-2}	henry/m	
Magnetic diffusivity	η	L^2T^{-1}	m^2/sec	$= 10^4$ cm^2/sec
Electric field strength	E	$MLT^{-2}Q^{-1}$	volt/m	$= 10^6$ e.m.u.
Magnetic field strength	H	$L^{-1}T^{-1}Q$	A·turn/m	$= 4\pi \times 10^{-3}$ oersted (Oe)
Magnetic flux		$ML^2T^{-1}Q^{-1}$	weber (Wb)	$= 10^8$ maxwell (Mx)
Magnetic flux density	B	$MT^{-1}Q^{-1}$	Wb/m^2 tesla (T)	$= 10^4$ gauss (G) $= 10^4$ gauss (G)
Magnetic dipole moment	$T^{(1)}$	$L^2T^{-1}Q$	$A \cdot m^2$ joule/T	$= 10^3$ erg/gauss $= 10^3$ erg/gauss

APPENDIX 2

MULTIPOLE MOMENTS OF CURRENTS IN A SPHERE

A.2.1 The multipole expansion of the external magnetic field

Consider the magnetic flux density $\underset{\sim}{B}$ due to a current distribution $\underset{\sim}{j}$ in a volume V whose exterior, $\hat{V}$, is insulating. The magnetic vector potential at the point $\underset{\sim}{r}$ is *(Stratton, 1941, p. 234)*

$$\underset{\sim}{A}(\underset{\sim}{r}) = \frac{\mu}{4\pi}\int_V \frac{\underset{\sim}{j}(\underset{\sim}{\xi})}{|\underset{\sim}{r}-\underset{\sim}{\xi}|}\, d\underset{\sim}{\xi} \tag{A2.1}$$

For points in $\hat{V}$, the quantity $|\underset{\sim}{r}-\underset{\sim}{\xi}|^{-1}$ in the integrand of (A2.1) may usefully be expanded in a Taylor series to give

$$\underset{\sim}{A}(\underset{\sim}{r}) = \frac{\mu}{4\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left\{\frac{\partial^m}{\partial x_{a_1}\partial x_{a_2}\cdots\partial x_{a_m}}\frac{1}{r}\right\}\cdot \cdot\int_V \xi_{a_1}\xi_{a_2}\cdots\xi_{a_m}\,\underset{\sim}{j}(\underset{\sim}{\xi})\,d\underset{\sim}{\xi} \tag{A2.2}$$

$$= \frac{\mu}{4\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_V\left\{(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^m\frac{1}{r}\right\}\underset{\sim}{j}(\underset{\sim}{\xi})\,d\underset{\sim}{\xi} \tag{A2.3}$$

The integrand in equation (A2.3) may be expanded as follows

$$\left\{(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^m\frac{1}{r}\right\}\underset{\sim}{j} = \frac{1}{(m+1)}\left\{\sum_{i=0}^{m}\underset{\sim}{\xi}^i\,\underset{\sim}{j}\,\underset{\sim}{\xi}^{m-i}\cdot\underset{\sim}{\nabla}^m\frac{1}{r} + + \sum_{i=1}^{m}(\underset{\sim}{j}\,\underset{\sim}{\xi}^m - \underset{\sim}{\xi}^i\,\underset{\sim}{j}\,\underset{\sim}{\xi}^{m-i})\cdot\underset{\sim}{\nabla}^m\frac{1}{r}\right\} \tag{A2.4}$$

where we have used the notation

$$\underset{\sim}{\xi}^{i}\,\underset{\sim}{j}\,\underset{\sim}{\xi}^{m-i}\cdot\underset{\sim}{\nabla}^{m} \equiv \tag{A2.5}$$

$$\equiv\ \underset{\sim}{1}_{k}\,\xi_{k}\,\xi_{a_1}\cdots\xi_{a_i}\,j_{a_{i+1}}\,\xi_{a_{i+2}}\cdots\xi_{a_m}\,\frac{\partial^{m}}{\partial x_{a_1}\cdots\partial x_{a_m}}$$

From (A2.4) we see that

$$\left\{(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^{m}\frac{1}{r}\right\}\underset{\sim}{j}\ =$$

$$=\frac{1}{(m+1)}\left\{(\underset{\sim}{j}\cdot\underset{\sim}{\nabla}')\underset{\sim}{\xi}^{m+1}\cdot\underset{\sim}{\nabla}^{m}\frac{1}{r}+\sum_{i=1}^{m}\left[\underset{\sim}{j}(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^{m}-\underset{\sim}{\xi}(\underset{\sim}{j}\cdot\underset{\sim}{\nabla})(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^{m-1}\right]\frac{1}{r}\right\}$$

$$=\frac{1}{(m+1)}\left\{\left[\underset{\sim}{\nabla}'\cdot(\underset{\sim}{j}\,\underset{\sim}{\xi}^{m+1})-(\underset{\sim}{\nabla}'\cdot\underset{\sim}{j})\underset{\sim}{\xi}^{m+1}\right]\cdot\underset{\sim}{\nabla}^{m}\frac{1}{r}\ +\right.$$

$$\left.+\ m\,(\underset{\sim}{j}\,\underset{\sim}{\xi}-\underset{\sim}{\xi}\,\underset{\sim}{j})(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^{m-1}\cdot\underset{\sim}{\nabla}\frac{1}{r}\ \right\}$$

$$=\frac{1}{(m+1)}\left\{\left[\underset{\sim}{\nabla}'\cdot(\underset{\sim}{j}\,\underset{\sim}{\xi}^{m+1})-(\underset{\sim}{\nabla}'\cdot\underset{\sim}{j})\underset{\sim}{\xi}^{m+1}\right]\cdot\underset{\sim}{\nabla}^{m}\frac{1}{r}\right\}\ + \tag{A2.6}$$

$$+\ \frac{m}{(m+1)}\left\{(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^{m-1}(\underset{\sim}{\xi}\times\underset{\sim}{j})\times\underset{\sim}{\nabla}\frac{1}{r}\right\}$$

where

$$\underset{\sim}{\nabla}'\ \equiv\ \underset{\sim}{1}_{i}\frac{\partial}{\partial\xi_{i}} \tag{A2.7}$$

$$\underset{\sim}{\nabla}\ \equiv\ \underset{\sim}{1}_{i}\frac{\partial}{\partial r_{i}}\ \equiv\ \underset{\sim}{1}_{i}\frac{\partial}{\partial x_{i}} \tag{A2.7'}$$

Substituting (A2.6) into (A2.3), and applying *Gauss' theorem*, we obtain

$$\underset{\sim}{A}(\underset{\sim}{r})\ =\frac{\mu}{4\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m}}{m!}\left\{\frac{1}{(m+1)}\left[\int_{S}\underset{\sim}{n}\cdot\underset{\sim}{j}\,\underset{\sim}{\xi}^{m+1}\,dS\ +\right.\right. \tag{A2.8}$$

$$\left.-\int_{V}(\underset{\sim}{\nabla}'\cdot\underset{\sim}{j})\,\underset{\sim}{\xi}^{m+1}d\underset{\sim}{\xi}\ \right]\cdot\underset{\sim}{\nabla}^{m}\frac{1}{r}\ +$$

$$\left.+\ \frac{m}{(m+1)}\int_{V}(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})^{m-1}(\underset{\sim}{\xi}\times\underset{\sim}{j})\times\underset{\sim}{\nabla}\frac{1}{r}\,d\underset{\sim}{\xi}\ \right\}$$

where S is the surface bounding the volume V . On S the boundary condition (1.59) applies - i.e.

$$\underset{\sim}{n} \cdot \underset{\sim}{j} = 0 \quad \text{on} \quad S \tag{A2.9}$$

Also, in the quasi-steady approximation equation (1.15) implies that

$$\underset{\sim}{\nabla}' \cdot \underset{\sim}{j} = 0 \tag{A2.10}$$

in V . Equation (A2.8) thus reduces to

$$\underset{\sim}{A}(\underset{\sim}{r}) = \frac{\mu}{4\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} T^{(m)} \cdot \underset{\sim}{\nabla}^{m-1} \times \underset{\sim}{\nabla} \frac{1}{r} \tag{A2.11}$$

where

$$T^{(m)} \equiv \frac{m}{(m+1)} \int_V (\underset{\sim}{\xi} \times \underset{\sim}{j}) \, \underset{\sim}{\xi}^{m-1} \, d\underset{\sim}{\xi} \tag{A2.12}$$

and we have used the notation

$$T^{(m)} \cdot \underset{\sim}{\nabla}^{m-1} \times \underset{\sim}{\nabla} \frac{1}{r} \equiv \tag{A2.13}$$

$$\equiv \underset{\sim}{1}_i \, \epsilon_{i a_1 k} \, T^{(m)}_{a_1 \cdots a_m} \frac{\partial^m}{\partial x_k \partial x_{a_2} \cdots \partial x_{a_m}} \frac{1}{r}$$

The magnetic flux density $\underset{\sim}{B}$ in $\hat{V}$, the *exterior* of volume V , is given by the *curl* of equation (A2.11).

$$\underset{\sim}{B}(\underset{\sim}{r}) = -\frac{\mu}{4\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left\{ T^{(m)} \cdot \underset{\sim}{\nabla}^m \right\} \underset{\sim}{\nabla} \frac{1}{r} \tag{A2.14}$$

Since $\underset{\sim}{B}$ is continuous across S , by equations (1.56) and (1.57), equation (A2.14) is also valid on S .

A.2.2 The magnetic multipole moments of a spherical current distribution

From equation (1.15) we have that

$$\boldsymbol{j} = \frac{1}{\mu}\,\boldsymbol{\nabla}' \times \boldsymbol{B} \qquad \text{(A2.15)}$$

in V . Therefore

$$\begin{aligned}
(\boldsymbol{\xi}\times\boldsymbol{j})\,\boldsymbol{\xi}^{m-1} &= \frac{1}{\mu}\left\{\boldsymbol{\xi}\times(\boldsymbol{\nabla}'\times\boldsymbol{B})\right\}\boldsymbol{\xi}^{m-1} \\
&= \frac{1}{\mu}\left\{\boldsymbol{\nabla}'(\boldsymbol{B}\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}'\boldsymbol{B} - \boldsymbol{B}\right\}\boldsymbol{\xi}^{m-1} \\
&= \frac{1}{\mu}\Big\{\boldsymbol{\nabla}'(\boldsymbol{B}\cdot\boldsymbol{\xi}^{m}) - (\boldsymbol{B}\cdot\boldsymbol{\xi})\,\boldsymbol{\nabla}'\boldsymbol{\xi}^{m-1} + \qquad \text{(A2.16)} \\
&\qquad - \boldsymbol{\nabla}'\cdot(\boldsymbol{\xi}\boldsymbol{B}\boldsymbol{\xi}^{m-1}) + (m+1)\,\boldsymbol{B}\boldsymbol{\xi}^{m-1}\Big\}
\end{aligned}$$

where we have made use of the identity

$$\begin{aligned}
&\boldsymbol{\nabla}'\cdot\left\{\boldsymbol{\xi}^{i}\boldsymbol{B}\boldsymbol{\xi}^{m-i}\right\} \qquad (i \geq 1) \\
&\equiv 3\,\boldsymbol{\xi}^{i-1}\boldsymbol{B}\boldsymbol{\xi}^{m-i} + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla}')\left\{\boldsymbol{\xi}^{i-1}\boldsymbol{B}\boldsymbol{\xi}^{m-i}\right\} \\
&\equiv 3\,\boldsymbol{\xi}^{i-1}\boldsymbol{B}\boldsymbol{\xi}^{m-i} + \boldsymbol{\xi}^{i-1}(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}'\boldsymbol{B})\,\boldsymbol{\xi}^{m-i} + (m-1)\,\boldsymbol{\xi}^{i-1}\boldsymbol{B}\boldsymbol{\xi}^{m-i} \\
&\equiv (m+2)\,\boldsymbol{\xi}^{i-1}\boldsymbol{B}\boldsymbol{\xi}^{m-i} + \boldsymbol{\xi}^{i-1}(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}'\boldsymbol{B})\,\boldsymbol{\xi}^{m-i} \qquad \text{(A2.17)}
\end{aligned}$$

Substituting (A2.16) into (A2.12), we obtain

$$\begin{aligned}
\frac{\mu(m+1)}{m}\,T^{(m)} &= \int_V \Big\{\boldsymbol{\nabla}'(\boldsymbol{B}\cdot\boldsymbol{\xi}^{m}) - \boldsymbol{\nabla}'(\boldsymbol{\xi}\boldsymbol{B}\boldsymbol{\xi}^{m-1}) + \\
&\qquad + (m+1)\,\boldsymbol{B}\boldsymbol{\xi}^{m-1} - (\boldsymbol{B}\cdot\boldsymbol{\xi})\,\boldsymbol{\nabla}'\boldsymbol{\xi}^{m-1}\Big\}\,d\boldsymbol{\xi} \\
&= \int_S \left\{\boldsymbol{n}\,\boldsymbol{B}\cdot\boldsymbol{\xi}^{m} - \boldsymbol{n}\cdot\boldsymbol{\xi}\boldsymbol{B}\boldsymbol{\xi}^{m-1}\right\} dS \;+ \qquad \text{(A2.18)} \\
&\qquad + \int_V \left\{(m+1)\,\boldsymbol{B}\boldsymbol{\xi}^{m-1} - (\boldsymbol{B}\cdot\boldsymbol{\xi})\,\boldsymbol{\nabla}'\boldsymbol{\xi}^{m-1}\right\} d\boldsymbol{\xi}
\end{aligned}$$

When V is a spherical volume,

$$\underset{\sim}{n} \equiv \left\{ \underset{\sim}{\xi}/|\underset{\sim}{\xi}| \right\}_S \tag{A2.19}$$

and

$$\underset{\sim}{n}\,\underset{\sim}{B}\cdot\underset{\sim}{\xi}^m = \underset{\sim}{n}\cdot\underset{\sim}{B}\,\underset{\sim}{\xi}^m \quad \text{on } S \tag{A2.20}$$

Also,

$$\underset{\sim}{n}\cdot\underset{\sim}{\xi}\,\underset{\sim}{B}\,\underset{\sim}{\xi}^{m-1} = r_o^m\,\underset{\sim}{B}\,\underset{\sim}{n}^{m-1} \quad \text{on } S \tag{A2.21}$$

where we have defined

$$r_o \equiv |\underset{\sim}{\xi}| \quad \text{on } S \tag{A2.22}$$

Substituting (A2.20) and (A2.21) into equation (A2.18), we obtain

$$\frac{\mu(m+1)}{m}\,T^{(m)} =$$

$$= -r_o^m \int_S \underset{\sim}{B}\,\underset{\sim}{n}^{m-1}\,dS + \int_S \underset{\sim}{n}\cdot\underset{\sim}{B}\,\underset{\sim}{\xi}^{m-1}\,d\underset{\sim}{S} +$$

$$+ \int_V \left\{ (m+1)\,\underset{\sim}{B}\,\underset{\sim}{\xi}^{m-1} - (\underset{\sim}{B}\cdot\underset{\sim}{\xi})\,\underset{\sim}{\nabla}'\underset{\sim}{\xi}^{m-1} \right\} d\underset{\sim}{\xi}$$

$$= -r_o^m \int_S \underset{\sim}{B}\,\underset{\sim}{n}^{m-1}\,dS + \int_V \underset{\sim}{\nabla}'\cdot(\underset{\sim}{B}\,\underset{\sim}{\xi}^{m-1})\,d\underset{\sim}{\xi} +$$

$$+ \int_V \left\{ (m+1)\,\underset{\sim}{B}\,\underset{\sim}{\xi}^{m-1} - (\underset{\sim}{B}\cdot\underset{\sim}{\xi})\,\underset{\sim}{\nabla}'\underset{\sim}{\xi}^{m-1} \right\} d\underset{\sim}{\xi}$$

$$= -r_o^m \int_S \underset{\sim}{B}\,\underset{\sim}{n}^{m-1}\,dS +$$

$$+ \int_V \left\{ \underset{\sim}{B}\cdot\underset{\sim}{\nabla}'\underset{\sim}{\xi}^m + (m+1)\,\underset{\sim}{B}\,\underset{\sim}{\xi}^{m-1} - (\underset{\sim}{B}\cdot\underset{\sim}{\xi})\,\underset{\sim}{\nabla}'\underset{\sim}{\xi}^{m-1} \right\} d\underset{\sim}{\xi}$$

$$= -r_o^m \int_S \underset{\sim}{B}\,\underset{\sim}{n}^{m-1}\,dS + \tag{A2.23}$$

$$+ \int_V \left\{ (m+2)\,\underset{\sim}{B}\,\underset{\sim}{\xi}^{m-1} + \underset{\sim}{\xi}\,\underset{\sim}{B}\cdot\underset{\sim}{\nabla}'\underset{\sim}{\xi}^{m-1} - (\underset{\sim}{B}\cdot\underset{\sim}{\xi})\,\underset{\sim}{\nabla}'\underset{\sim}{\xi}^{m-1} \right\} d\underset{\sim}{\xi}$$

since

$$\underset{\sim}{\nabla}' \cdot \underset{\sim}{B} = 0 \tag{A2.24}$$

by equation (1.3).

A.2.3 Surface terms, and theorems on spherical harmonics

Since the expansion (A2.14) is valid on S , we may substitute it into the surface integrals in equation (A2.23). Carrying out this substitution, and making use of the notation (5.93)-(5.95), we obtain

$$\int_S \underset{\sim}{B}\, \underset{\sim}{n}^{m-1}\, dS = \tag{A2.25}$$

$$= -\frac{\mu}{4\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} T^{(k)}_{a_1 \cdots a_k} \int_S \partial_{a_1} \cdots \partial_{a_k} \underset{\sim}{\nabla} \frac{1}{r}\, \underset{\sim}{n}^{m-1}\, dS$$

In order to evaluate the integrals on the right hand side of (A2.25), we must consider the properties of *spherical harmonics*.

Since the differential operators $\partial_{a_i} \equiv \partial/\partial x_{a_i}$ and $\underset{\sim}{\nabla}$ are commutative,

$$\nabla^2 \partial_{a_1} \cdots \partial_{a_k} \frac{1}{r} = \partial_{a_1} \cdots \partial_{a_k} \nabla^2 \frac{1}{r} = 0 \tag{A2.26}$$

Thus $\partial_{a_1} \cdots \partial_{a_k} \frac{1}{r}$ is a *spherical harmonic*. In general, for any polynomial homogeneous function $f_n(x,y,z)$ of degree n , $f_n(\partial_x, \partial_y, \partial_z)\frac{1}{r}$ is a *spherical harmonic of*

degree -(n+1), or else is zero (Hobson, 1931, p. 127). The expression vanishes if and only if $f_n(x,y,z)$ is a multiple of $\{x^2 + y^2 + z^2\}$. It follows that $\partial_{a_1} \cdots \partial_{a_k} \frac{1}{r}$ *is a spherical harmonic of degree -(k+1).* From the properties of spherical harmonics, we have that

$$\underset{\sim}{n} \cdot \underset{\sim}{\nabla}\, \partial_{a_1} \cdots \partial_{a_k} \frac{1}{r} = \frac{\partial}{\partial r} \partial_{a_1} \cdots \partial_{a_k} \frac{1}{r}$$

$$= -\frac{(k+1)}{r_o} \partial_{a_1} \cdots \partial_{a_k} \frac{1}{r} \qquad \text{(A2.27)}$$

on S, as pointed out in equation (5.96).

It may also be shown *(Hobson, 1931, pp. 147-148)* that the components $\{n_i\}$ of the unit vector normal to S satisfy

$$n_{b_1} \cdots n_{b_q} = \frac{1}{r^q} x^{p_1} y^{p_2} z^{p_3} \quad , \text{ where } (p_1 + p_2 + p_3) = q$$

$$= \sum_{i=0}^{[q/2]} Y_{q-2i}(r; \theta, \phi) \qquad \text{(A2.28)}$$

where $[q/2]$ denotes the largest integer less than or equal to $q/2$, and $Y_n(r;\theta,\phi)$ is a solid spherical harmonic of degree n. It follows that the integrals appearing on the right hand side of equation (A2.25) are of the form

$$\int_S \partial_{a_1} \cdots \partial_{a_k} \frac{1}{r}\, n_{b_1} \cdots n_{b_m}\, dS =$$

$$= \frac{1}{r_o^k} \sum_{i=0}^{[m/2]} \int_S Y_k' \, Y_{q-2i}\, dS \qquad \text{(A2.29)}$$

where Y_k' and Y_{q-2i} are spherical harmonics of degrees n and $q-2i$ respectively.

In general *(Hobson, 1931, p. 144)*, if Y'_m and Y_n are spherical harmonics of degrees m and n respectively,

$$\int_S Y'_m Y_n \, dS = 0 \quad , \quad m \neq n \tag{A2.30}$$

Combining (A2.29) and (A2.30), and substituting the resulting expression into equation (A2.25), we find that

$$\int_S \underset{\sim}{B} \, \underset{\sim}{n}^{m-1} dS =$$

$$= -\frac{\mu}{4\pi} \sum_{k=1}^{[(m-1)/2]} \frac{(-1)^{m+1}}{(m-2k)!} T^{(m-2k)}_{a_1 \cdots a_{m-2k}} \cdot \tag{A2.31}$$

$$\cdot \int_S \partial_{a_1} \cdots \partial_{a_{m-2k}} \partial_{b_1} \frac{1}{r} \, n_{b_2} \cdots n_{b_m} \underset{\sim}{1}_{b_1} \cdots \underset{\sim}{1}_{b_m}$$

In order to obtain explicit expressions for the integrals on the right hand side of (A2.31) we must apply the theorems

$$\int_S n_{b_1} \cdots n_{b_q} \, dS = 4\pi r_0^2 \, \frac{2^{q/2} (q/2)!}{(q+1)!} \sum_{\substack{\text{pairs} \\ (b_1 \cdots b_q)}} \delta_{\gamma_1 \gamma_2} \cdots \delta_{\gamma_{q-1} \gamma_q} \, ,$$

$$(q \text{ even}) \tag{A2.32}$$

$$\int_S n_{b_1} \cdots n_{b_q} \, dS = 0 \quad , \quad (q \text{ odd}) \tag{A2.32'}$$

$$\partial_{a_1} \cdots \partial_{a_m} \frac{1}{r} = \frac{(-1)^m}{r^{m+1}} \sum_{i=0}^{[m/2]} (-1)^i \frac{(2m-2i)!}{2^{m-i} (m-i)!} \cdot \tag{A2.33}$$

$$\cdot \sum_{\substack{(a_1 \cdots a_m) \\ (\epsilon_{2i} | \eta_{m-2i})}} n_{\eta_1 \eta_2 \cdots \eta_{m-2i}} \sum_{\substack{\text{pairs} \\ (\epsilon_1 \cdots \epsilon_{2i})}} \delta_{\gamma_1 \gamma_2} \cdots \delta_{\gamma_{2i-1} \gamma_{2i}}$$

where

$\sum\limits_{\substack{(a_1\ldots a_m)\\(\epsilon_{2i}|\eta_{m-2i})}}$ *denotes a summation over all* ***distinct*** *partitions of the set* $(a_1,a_2,\ldots,a_m)$ *into one subset of* $2i$ *elements and another of* $m-2i$ *elements,*

and

$\sum\limits_{\substack{\text{pairs}\\(c_1\ldots c_{2i})}}$ *denotes a summation over all* ***distinct*** *groupings of the elements of the set* $(c_1,c_2,\ldots,c_{2i})$ *into* ***pairs*** $(\gamma_1\gamma_2;\ \gamma_3\gamma_4;\ldots;\ \gamma_{2i-1}\gamma_{2i})$.

Equations (A2.32) and (A2.32') can be verified from the formula given on *p. 156 of Hobson (1931)*. Equation (A2.33) can be verified by induction.

Substituting (A2.32)-(A2.33) into equation (A2.31), we obtain the expression

$$r_o^n \int_S \underset{\sim}{B}\, \underset{\sim}{n}^{n-1}\, dS =$$

$$= 2\mu \sum_{k=1}^{[(n-1)/2]} \frac{2^k r_o^{2k}}{(n-2k)!}\, T^{(n-2k)}_{a_1\ldots a_{n-2k}} \cdot \underset{\sim}{1}_{b_1}\cdots\underset{\sim}{1}_{b_n}\ \cdot$$

$$\cdot \sum_{i=0}^{\left[\frac{n+1}{2}-k\right]} (-1)^i \frac{(2n-4k-2i+2)!\,(n-k-i)!}{(2n-2k-2i+1)!\,(n-2k-i+1)!}\ \cdot \tag{A2.34}$$

$$\cdot \sum_{\substack{(a_1\ldots a_{n-2k},\, b_1)\\(\epsilon_{2i}|\eta_{n-2k+1-2i})}} \sum_{\substack{\text{pairs}\\(\eta_1\cdots\eta_{n-2k+1-2i},\, b_2\cdots b_n)}} \delta_{\zeta_1\zeta_2}\cdots\delta_{\zeta_{2n-2k-2i-1}\zeta_{2n-2k-2i}}\ \cdot$$

$$\cdot \sum_{\substack{\text{pairs}\\(\epsilon_1\cdots\epsilon_{2i})}} \delta_{\gamma_1\gamma_2}\cdots\delta_{\gamma_{2i-1}\gamma_{2i}}$$

A.2.4 Expressions for the magnetic multipole moments

Equation (A2.34) may now be substituted into equation (A2.23) to give the following expression for the magnetic multipole moments of a current distribution in a sphere.

$$
\begin{aligned}
T^{(m)} = \; & \frac{m}{\mu(m+1)} \int_V \Big\{ (m+2) \underset{\sim}{B} \underset{\sim}{\xi}^{m-1} + \underset{\sim}{\xi} \underset{\sim}{B} \cdot \underset{\sim}{\nabla}' \underset{\sim}{\xi}^{m-1} + \\
& \qquad\qquad - (\underset{\sim}{B} \cdot \underset{\sim}{\xi}) \underset{\sim}{\nabla}' \underset{\sim}{\xi}^{m-1} \Big\} \, d\underset{\sim}{\xi} \; + \\
& - \frac{2m}{(m+1)} \sum_{k=1}^{\left[\frac{m-1}{2}\right]} \frac{2^k r_0^{2k}}{(m-2k)!} T^{(m-2k)}_{a_1 \cdots a_{m-2k}} \cdot \underset{\sim}{1}_{b_1} \cdots \underset{\sim}{1}_{b_m} \cdot \\
& \cdot \sum_{i=0}^{\left[\frac{m+1}{2} - k\right]} (-1)^i \frac{(2m-4k-2i+2)!\,(m-k-i)!}{(2m-2k-2i+1)!\,(m-2k-i+1)!} \cdot \\
& \cdot \sum_{\substack{(a_1 \cdots a_{m-2k}, b_1) \\ (\epsilon_{2i} \mid \eta_{m-2k+1-2i})}} \; \sum_{\substack{\text{pairs} \\ (\eta_1 \cdots \eta_{m-2k+1-2i}, b_2 \cdots b_m)}} \delta_{\zeta_1 \zeta_2} \cdots \delta_{\zeta_{2m-2k-2i-1} \zeta_{2m-2k-2i}} \cdot \\
& \cdot \sum_{\substack{\text{pairs} \\ (\epsilon_1 \cdots \epsilon_{2i})}} \delta_{\gamma_1 \gamma_2} \cdots \delta_{\gamma_{2i-1} \gamma_{2i}}
\end{aligned}
\qquad \text{(A2.35)}
$$

This expression is written out in full for $m = 1,2,3$ in equations (5.77)-(5.79).

It may be seen from equation (A2.35) that the expression for the 2^m-pole moment tensor $T^{(m)}_{a_1 a_2 \cdots a_m}$ involves terms of two different types. The first type of term, represented by the volume integrals in (A2.35), relates $T^{(m)}$ to the (m-1)-order integral moments of the

magnetic flux density B in the spherical volume V. The second type of term, represented by the summation terms in (A2.35), relates $T^{(m)}$ to the magnetic multipole tensors of order $\{[m-2], [m-4], \ldots, [2 \text{ or } 1]\}$. Thus $T^{(m)}$ has *explicit* relationships only with the *lower-order magnetic multipole moments of the same parity* (i.e. m even or odd). Relationships with multipole moments of the *opposite parity* will arise from the integral terms in equation (A2.35).

A.2.5 Electric multipole moments

The multipole expansion of the electric field is obtained in much the same way as that for the magnetic field. In general, the electric field $\underset{\sim}{E}$ satisfies

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\Phi - \partial\underset{\sim}{A}/\partial t \tag{A2.36}$$

In the *exterior region* $\hat{V}$, the vector potential $\underset{\sim}{A}$ is given by the expansion (A2.11), while the scalar potential Φ is given by

$$\Phi(\underset{\sim}{r}) = \frac{1}{4\pi\epsilon}\int_V \frac{\theta(\underset{\sim}{\xi})}{|\underset{\sim}{r}-\underset{\sim}{\xi}|}\,d\underset{\sim}{\xi}$$

$$= \frac{1}{4\pi\epsilon}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_V \theta(\underset{\sim}{\xi})\,\{\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\}^m\frac{1}{r}\,d\underset{\sim}{\xi}$$

$$= \frac{1}{4\pi\epsilon}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\Big\{\int_V \theta\,\underset{\sim}{\xi}^m d\underset{\sim}{\xi}\Big\}\cdot\underset{\sim}{\nabla}^m\frac{1}{r} \tag{A2.37}$$

The electric multipole moment tensors, $Q^{(m)}$, may therefore be defined as

$$\underset{\sim}{Q}^{(m)} \equiv \int_V \theta(\underset{\sim}{\xi}) \, \underset{\sim}{\xi}^m \, d\underset{\sim}{\xi} \tag{A2.38}$$

Substituting (A2.11), (A2.37), and (A2.38) into equation (A2.36), we obtain the expansion

$$\underset{\sim}{E} = -\frac{1}{4\pi\epsilon} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} Q^{(m)} \cdot \underset{\sim}{\nabla}^m \underset{\sim}{\nabla} \frac{1}{r} + \tag{A2.39}$$

$$- \frac{\mu}{4\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \dot{T}^{(m)} \cdot \underset{\sim}{\nabla}^{m-1} \times \underset{\sim}{\nabla} \frac{1}{r}$$

There is no contribution from the zero-order electric multipole moment tensor $Q^{(0)}$ in (A2.39) since, as shown in equation (5.75),

$$Q^{(0)} = -\epsilon \int_V \underset{\sim}{\nabla} \cdot \{ \underset{\sim}{u} \times \underset{\sim}{B} \} \, d\underset{\sim}{\xi}$$

$$= -\epsilon \int_S \underset{\sim}{n} \cdot \{ \underset{\sim}{u} \times \underset{\sim}{B} \} \, dS \tag{A2.40}$$

in the quasi-steady approximation. When the *no-slip condition* (1.62) applies on S , equation (A2.40) reduces to

$$Q^{(0)} = 0 \tag{A2.40'}$$

Equation (A2.39) is the expansion used for $\underset{\sim}{E}$ in $\hat{V}$ in *section 5.4.2.*

APPENDIX 3

EVALUATION OF INTEGRALS ASSOCIATED WITH INITIAL CONDITION *I* IN CHAPTER 3

A.3.1 Integrals of *section 3.7.4*

Many of the integrals associated with the mean field dispersion relation studied in *Chapter 3* must be evaluated numerically. In this appendix we shall deal with the integrals associated with *initial condition I*.

In *section 3.7.4* the dispersion relation for Gaussian isotropic turbulence and non-oscillatory mean fields is considered. An approximate form of the relation is obtained by assuming equality on the right hand side of condition (3.105) - i.e. assuming that

$$\mathcal{I}m\,\frac{\Omega}{\eta K^2} - 1 = (R_m')^2\, J_2 \qquad \text{(A3.1)}$$

where J_2 is defined in (3.107). The present author and *Dr. K.D. Aldridge* have developed a program which evaluates the integral in (A3.1), thus giving R_m' as an approximate function of $\text{Im}\;\Omega/\eta K^2$, q , $\lambda_c K$, and $T/q\tau_c$.

The required inputs to the program are:

$$Q \equiv q \qquad \text{(A3.2)}$$

$$\text{YNOT} \equiv \text{Im}\;\Omega/\eta K^2 \qquad \text{(A3.3)}$$

$$\text{ALP} \equiv \lambda_c/L = \lambda_c K/2\pi \qquad \text{(A3.4)}$$

$$\text{XUL} \equiv T/q\tau_c \qquad \text{(A3.5)}$$

NX $\equiv$ half the number of integration steps (A3.6)

The integration is carried out using a *Simpson's Rule* technique. The program outputs are:

ALP $\equiv$ λ_c/L

A1 $\equiv$ $J_2(q,\lambda_c K;T)$, defined in (3.107) (A3.7)

A2 $\equiv$ $[R_m'/q]^2$, defined by (A3.1) (A3.8)

A3 $\equiv$ R_m'/q (A3.9)

The integral in (3.107) is denoted by VOLR .

```
          PI=3.141592653
        2 WRITE(6,750)
      750 FORMAT('NEED Q,YNOT')
          READ(5,250,END=40)  Q,YNOT
      250 FORMAT(2F10.7)
        7 WRITE(6,930)
          READ(5,940,END=2)  ALP
        3 WRITE(6,850)
      850 FORMAT('NEED XUL,NX')
          READ(5,350,END=7)  XUL,NX
      350 FORMAT(1F10.7,I3)
      930 FORMAT('NEED ALP')
      940 FORMAT(F10.7)
          VOLR=SIMRE(ALP,Q,YNOT,XUL,NX)
          A1=VOLR/3.
          A2=(YNOT-1.)/(A1*Q*Q)
          A3=SQRT(A2)
       21 WRITE(6,920)  ALP,A1,A2,A3
          WRITE(6,970)
      970 FORMAT(//)
      920 FORMAT(F7.4,3E18.8)
          GO TO 3
       40 STOP
          END
```

(A3.10)

```
      FUNCTION ENV(ALP,Q,YNOT,X)
      PI=3.141592653
      GX=4.*PI*PI*ALP*ALP*X
      T=1./(1.+2.*X)
      EX=-X*X*Q*Q/2.+GX*(YNOT-T)
      IF(EX.GT.170) GO TO 40
      GO TO 60
   40 ENV=0
      GO TO 100
   60 ENV=EXP(EX)*T**2.5
      GO TO 300
  100 WRITE(6,200) EX
  200 FORMAT('EXPONENT=',F10.2)
  300 RETURN
      END
```

(A3.10')

```
      FUNCTION SIMRE(ALP,Q,YNOT,XUL,N)
      EXTERNAL ENV
      PI=3.141592653
      H=XUL/(2.*N)
      SUMEV=0
      SUMOD=0
      DO 49 J=1,N
      HOD=H*(2.*J-1)
      HEV=H*(2.*J-2.)
      SUMEV=SUMEV+ENV(ALP,Q,YNOT,HEV)
      SUMOD=SUMOD+ENV(ALP,Q,YNOT,HOD)
   49 CONTINUE
      FST=ENV(ALP,Q,YNOT,0)
      FFN=ENV(ALP,Q,YNOT,XUL)
      SUM=4.*SUMOD+2.*SUMEV-FST+FFN
      SIMRE=H*SUM/3.
      RETURN
      END
```

(A3.10")

A.3.2 Integrals of *section 3.8.5*

Section 3.8.5 deals with conditions on the turbulence for stable decay to be established before significant energy is lost from the mean field. It is shown that the *stabilization time* T_1 must be less than T^*, where T^* is defined by (3.132). Combining (3.132) and (3.133), we see that T^* may also be determined from the equation

$$\begin{aligned} \mathrm{DELT}\,(T/\tau_c) &\equiv \left\{ \mathrm{Im}\frac{\Omega}{\eta K^2}\int_0^{T/q\tau_c} e^{-\frac{1}{2}q^2x^2 + q\tau_c\eta K^2\left[\mathrm{Im}\frac{\Omega}{\eta K^2} - \frac{1}{1+2x}\right]x}\,\frac{dx}{(1+2x)^{5/2}} - \frac{1}{(\lambda_c K)^2} \right\}(\lambda_c K)^2 \\ &= 0 \end{aligned} \tag{A3.11}$$

The program given below finds approximate zeroes of DELT . The required inputs are:

$$\begin{aligned} \mathrm{Q} &\equiv q \\ \mathrm{ALP} &\equiv \lambda_c/L \\ \mathrm{YNOT} &\equiv \text{trial value of } \mathrm{Im}\,\Omega/\eta K^2 \\ \mathrm{XUL} &\equiv T/q\tau_c \\ \mathrm{NX} &\equiv \text{half number of integration steps} \\ \mathrm{TOL} &\equiv \text{upper limit on DELT for acceptable solution of (A3.11)} \end{aligned} \tag{A3.12}$$

Outputs from the program are

Q = q

ALP = λ_c/L

YNOT $\equiv$ value of $\mathrm{Im}\,\Omega/\eta K^2$ for which (A3.11) is approximately satisfied

T1 $\equiv$ value of T/τ_c for which (A3.11) is approximately satisfied

(A3.13)

A1 $\equiv$ value of R_m'/q defined by T1 and (3.134)

DELT $\equiv$ final value of DELT

It should be noted that (A3.11) is in fact a generalization of (3.132), since the upper limit of integration in (3.132) has been allowed to go to ∞.

```
          PI=3.141592653
        2 WRITE(6,750)
      750 FORMAT('NEED Q,ALP,YNOT,XUL,NX,TOL')
          READ(5,250,END=40)  Q,ALP,YNOT,XUL,NX,TOL
      250 FORMAT(4F10.7,I3,F10.7)
          TPA=2.*PI*ALP
       25 YP=YNOT+TOL*(YNOT-1.)
          YM=YNOT-TOL*(YNOT-1.)
          VOLP=SIMRE(ALP,Q,YP,XUL,NX)
          VOLM=SIMRE(ALP,Q,YM,XUL,NX)
          VOLR=SIMRE(ALP,Q,YNOT,XUL,NX)
          DELT=TPA*TPA*YNOT*VOLR-1.
          IF(ABS(DELT).LT.TOL)  GO TO 311
          DDEL=TPA*TPA*(VOLR-YNOT*(VOLP-VOLM)/(2.*TOL*
         1 (YNOT-1.)))
          IF(DDEL.EQ.0) GO TO 2
          YNOT=YNOT-DELT/DDEL
          IF(YNOT.LT.1.) GO TO 40
          GO TO 25
      311 T1=Q/(TPA*TPA*YNOT)
          A1=TPA*SQRT(3.*YNOT*(YNOT-1.))/Q
          WRITE(6,920) Q,ALP,YNOT,T1,A1,DELT
      920 FORMAT(2F7.4,F7.5,3E18.8,//)
          GO TO 2
       40 STOP
          END
```

(A3.14)

```
      FUNCTION ENV(ALP,Q,YNOT,X)
      PI=3.141592653
      GX=4.*PI*PI*ALP*ALP*X
      T=1./(1.+2.*X)
      EX=-X*X*Q*Q/2.+GX*(YNOT-T)
      IF(EX.GT.170) GO TO 40
      GO TO 60
   40 ENV=0
      GO TO 100
   60 ENV=EXP(EX)*T**2.5
      GO TO 300
  100 WRITE(6,200) EX
  200 FORMAT('EXPONENT=',F10.2)
  300 RETURN
      END
```

(A3.14')

```
      FUNCTION SIMRE(ALP,Q,YNOT,XUL,N)
      EXTERNAL ENV
      PI=3.141592653
      H=XUL/(2.*N)
      SUMEV=0
      SUMOD=0
      DO 49 J=1,N
      HOD=H*(2.*J-1)
      HEV=H*(2.*J-2.)
      SUMEV=SUMEV+ENV(ALP,Q,YNOT,HEV)
      SUMOD=SUMOD+ENV(ALP,Q,YNOT,HOD)
   49 CONTINUE
      FST=ENV(ALP,Q,YNOT,0)
      FFN=ENV(ALP,Q,YNOT,XUL)
      SUM=4.*SUMOD+2.*SUMEV-FST+FFN
      SIMRE=H*SUM/3.
      RETURN
      END
```

(A3.14")

A.3.3 Integrals of *section 3.9.2*

When the dispersion relation for Gaussian isotropic turbulence and oscillatory mean fields is considered, the integrals to be evaluated are those defined in *section 3.9.2*. The techniques of the last two sections can be used in the case when the *approximate* dispersion relation (3.143) is to be solved. The subroutines given below evaluate the integrals in (3.144a) and (3.144b) by a *Simpson's Rule* technique similar to that used in (A3.10"). The required inputs are:

$$\begin{aligned} \text{ALP} &\equiv \lambda_c/L \\ \text{BEP} &\equiv \tau_c \eta K^2/2\pi \\ \text{ZNOT} &\equiv \text{Re}\ \Omega/\eta K^2 \\ \text{YNOT} &\equiv \text{Im}\ \Omega/\eta K^2 \\ \text{N} &\equiv \text{half number of integration points} \end{aligned} \tag{A3.15}$$

and the outputs are:

$$\begin{aligned} \text{SIMRE} &\equiv 3J_{2R}\text{, defined in (3.144a)} \\ \text{SIMIM} &\equiv 3J_{2I}\text{, defined in (3.144b)} \end{aligned} \tag{A3.16}$$

The subroutine ENV is included for completeness.

No outline is given here of the methods used in searching for solutions to (3.143a) and (3.143b). See *section 3.9.4* and *Appendix 4*.

```
      FUNCTION SIMRE(ALP,BEP,ZNOT,YNOT,XUL,N,K)
      EXTERNAL ENV
      PI=3.141592653
      H=XUL/(2.*N)
      SUMEV=0
      SUMOD=0
      DO 49 J=1,N
      HOD=H*(2.*J-1)
      HEV=H*(2.*J-2.)
      COSEV=COS(2.*PI*BEP*ZNOT*HEV)
      SUMEV=SUMEV+ENV(ALP,BEP,ZNOT,YNOT,HEV,K)*COSEV
      COSOD=COS(2.*PI*BEP*ZNOT*HOD)
      SUMOD=SUMOD+ENV(ALP,BEP,ZNOT,YNOT,HOD,K)*COSOD
   49 CONTINUE
      FST=ENV(ALP,BEP,ZNOT,YNOT,0,K)
      COSLST=COS(2.*PI*BEP*ZNOT*XUL)
      FFN=ENV(ALP,BEP,ZNOT,YNOT,XUL,K)*COSLST
      SUM=4.*SUMOD+2*SUMEV-FST+FFN
      SIMRE=H*SUM/3.
      RETURN
      END
```

(A3.17)

```
      FUNCTION SIMIM(ALP,BEP,ZNOT,YNOT,XUL,N)
      EXTERNAL ENV
      PI=3.141592653
      H=XUL/(2.*N)
      SUMEV=0
      SUMOD=0
      DO 49 J=1,N
      HOD=H*(2.*J-1)
      HEV=H*(2.*J-2.)
      SINEV=SIN(2.*PI*BEP*ZNOT*HEV)
      SUMEV=SUMEV+ENV(ALP,BEP,ZNOT,YNOT,HEV)*SINEV
      SINOD=SIN(2.*PI*BEP*ZNOT*HOD)
      SUMOD=SUMOD+ENV(ALP,BEP,ZNOT,YNOT,HOD)*SINOD
   49 CONTINUE
      FST=0.
      SINLST=SIN(2.*PI*BEP*ZNOT*XUL)
      FFN=ENV(ALP,BEP,ZNOT,YNOT,XUL)*SINLST
      SUM=4.*SUMOD+2*SUMEV-FST+FFN
      SIMIM=H*SUM/3.
      RETURN
      END
```

(A3.17')

```
      FUNCTION ENV(ALP,BEP,ZNOT,YNOT,X,N)
      BEPX=BEP*X
      PI=3.141592653
      PALPSQ=PI*ALP*ALP
      T=PALPSQ/(PALPSQ+BEPX)
      EX=-X*X/2.+2.*PI*BEPX*(YNOT-T)
      IF(EX.GT.170) GO TO 40
      GO TO 60
   40 ENV=0
      GO TO 100
   60 IF(N.EQ.0) ENV=EXP(EX)*T**2.5
      IF(N.GT.0) ENV=EXP(EX)*T**2.5*X**N
      GO TO 300
  100 WRITE(6,200) EX
  200 FORMAT('EXPONENT=',F10.2)
  300 RETURN
      END
```

(A3.17")

APPENDIX 4

EVALUATION OF INTEGRALS ASSOCIATED WITH INITIAL CONDITION *II* IN CHAPTER 3

A.4.1 The integrals of *section 3.10.1*

In this appendix we shall consider the numerical solution of equations (3.165a) and (3.165b). The program described here was developed by the present author and *Dr. K.D. Aldridge* to obtain the results plotted in *Figures 16-21.*

The integrals in equations (3.165a,b), corresponding to the case of Gaussian turbulence in an incompressible fluid, are not in the most convenient form for numerical evaluation. We shall first obtain an explicit expression for Θ . Combining equations (3.42) and (3.43), and carrying out the integration, we have

$$\Theta = 2\xi^4 \int_0^{\pi} \frac{\sin^3\theta \, d\theta}{(1+\xi^2 - p + 2\xi\cos\theta) + i\nu}$$

$$= \xi^3 \int_{-1}^{1} \frac{(1-x^2)\,dx}{X + x}$$

$$= 2\xi^3 \left\{ X + (X^2-1)\cdot \tfrac{1}{2} \ln \frac{X-1}{X+1} \right\} \tag{A4.1}$$

where

$$x \equiv \cos\theta \tag{A4.2}$$

$$X \equiv \frac{1}{2\xi}\left\{ 1 + \xi^2 - p + i\nu \right\} \tag{A4.3}$$

Defining

$$F \equiv 1 + \xi^2 - p \tag{A4.4}$$

so that

$$X = \frac{1}{2\xi}\{F + i\nu\} \tag{A4.5}$$

we may take real and imaginary parts of (A4.1) to obtain

$$\mathcal{R}e\,\Theta = F\xi^2 + \frac{1}{8}\xi(F^2 - \nu^2 - 4\xi^2)\ln\left\{\frac{(F-2\xi)^2+\nu^2}{(F+2\xi)^2+\nu^2}\right\} + - \frac{1}{2}F\xi\nu\tan^{-1}\left\{\frac{4\xi\nu}{F^2-4\xi^2+\nu^2}\right\} \tag{A4.6a}$$

$$\mathcal{I}m\,\Theta = \nu\xi^2 + \frac{1}{4}F\nu\ln\left\{\frac{(F-2\xi)^2+\nu^2}{(F+2\xi)^2+\nu^2}\right\} + + \frac{1}{4}\xi(F^2-\nu^2-4\xi^2)\tan^{-1}\left\{\frac{4\xi\nu}{F^2-4\xi^2+\nu^2}\right\} \tag{A4.6b}$$

Further simplification of the integrals in (3.165a,b) can be obtained by making use of the fact that $\mathrm{Re}\,\Theta$ is *even* in ν, while $\mathrm{Im}\,\Theta$ is *odd* in ν. Thus if

$$I^{(3.165)} \equiv \int_{-\infty}^{\infty} d\nu \int_0^{\infty} d\xi \; e^{-\frac{1}{2}\left\{(\lambda_c K\xi)^2 + (\eta K^2\tau_c)^2(\nu + \mathcal{R}e\frac{\Omega}{\eta K^2})^2\right\}} \cdot \cdot\, \Theta(\xi, \nu; \mathcal{I}m\frac{\Omega}{\eta K^2}) \tag{A4.7}$$

we may write

$$\mathcal{R}e\, I^{(3.165)} = 2\int_0^{\infty} d\nu \int_0^{\infty} d\xi \;\; \mathcal{R}e\,\Theta \; \cosh\{\sqrt{2}\,\tau_c \mathcal{R}e\Omega\cdot\nu\} \cdot \cdot\, e^{-\frac{1}{2}\left\{(\lambda_c K\xi)^2 + (\eta K^2\tau_c)^2(\nu^2 + [\mathcal{R}e\frac{\Omega}{\eta K^2}]^2)\right\}}$$

(A4.8)

$$\mathcal{I}m\, I^{(3.165)} = 2\int_0^{\infty} d\nu \int_0^{\infty} d\xi \;\; \mathcal{I}m\,\Theta \; \sinh\{\sqrt{2}\,\tau_c \mathcal{R}e\Omega\cdot\nu\} \cdot \cdot\, e^{-\frac{1}{2}\left\{(\lambda_c K\xi)^2 + (\eta K^2\tau_c)^2(\nu^2 + [\mathcal{R}e\frac{\Omega}{\eta K^2}]^2)\right\}}$$

or

$$\mathcal{R}e\, I^{(3.165)} = \tag{A4.9a}$$

$$= \int_0^\infty d\nu \int_0^\infty d\xi \;\mathcal{R}e\, \Theta \left\{ e^{-\frac{1}{2}\left[(\lambda_c K\xi)^2 + (\nu + \mathcal{R}e\frac{\Omega}{\eta K^2})^2\right]} + \right.$$
$$\left. + \; e^{-\frac{1}{2}\left[(\lambda_c K\xi)^2 + (\nu - \mathcal{R}e\frac{\Omega}{\eta K^2})^2\right]} \right\}$$

$$\mathcal{I}m\, I^{(3.165)} = \tag{A4.9b}$$

$$= \int_0^\infty d\nu \int_0^\infty d\xi \;\mathcal{I}m\, \Theta \left\{ e^{-\frac{1}{2}\left[(\lambda_c K\xi)^2 + (\nu + \mathcal{R}e\frac{\Omega}{\eta K^2})^2\right]} + \right.$$
$$\left. - \; e^{-\frac{1}{2}\left[(\lambda_c K\xi)^2 + (\nu - \mathcal{R}e\frac{\Omega}{\eta K^2})^2\right]} \right\}$$

Equations (A4.9a,b) show the form of the integrals in (3.165a,b) used in the program.

A.4.2 The nature of the program

A brief outline of the numerical search procedure used is given in *section 3.10.2.* The integrations in (A4.9a,b) are carried out using an n-point Gaussian technique. The first input required by the program is a listing of the Gaussian points and weights to be used. A convenient form for the storage of this data is shown in *section A.4.4.*

The next set of inputs required is listed on *line 370* of the program *(see section A.4.3).* For example,

EPY1 $\equiv$ fraction of the peak value of the integrand in (A4.9a,b) below which contributions from the region near $\xi = 0$ can be ignored

Similar definitions apply to EPY4 , EPG1 , and EPG2 .

These quantities refer to the regions $\xi \to \infty$, $\nu \to 0$, and $\nu \to \infty$, respectively. In most cases it was assumed that

$$\mathrm{EPY1} = \mathrm{EPY4} = \mathrm{EPG1} = \mathrm{EPG2} = 0.0001 \qquad \text{(A4.10)}$$

The last input requested in *line 370* - the "number of steps, K" - is a measure of the "fineness" of the search for the point at which the value of the integrand drops to a fraction EPY(i) or EPG(i) of its peak value. In general, it was assumed that

$$K = 80 \qquad \text{(A4.10')}$$

The remaining inputs are as follows:

$$\left.\begin{aligned}
\mathrm{ZNOT} &\equiv \mathrm{Re}\ \Omega/\eta K^2 \\
\mathrm{YNOT} &\equiv \mathrm{Im}\ \Omega/\eta K^2 \\
\mathrm{STALF} &\equiv \text{initial value of } \lambda_c/L \\
\mathrm{ALFSTP} &\equiv \text{increment in } \lambda_c/L \\
\mathrm{IALF} &\equiv \text{number of steps in } \lambda_c/L \\
\mathrm{STBET} &\equiv \text{initial value of } \tau_c\eta K^2/2\pi \\
\mathrm{BETSTP} &\equiv \text{increment in } \tau_c\eta K^2/2\pi \\
\mathrm{IBET} &\equiv \text{number of steps in } \tau_c\eta K^2/2\pi
\end{aligned}\right\} \qquad \text{(A4.11)}$$

For convenience in calculation, λ_c/L and $\tau_c\eta K^2/2\pi$ values are multiplied by a factor $\sqrt{2}\cdot\pi$ in *lines 480-520*. The program then works with the quantities

$$\mathrm{ALFA} = \mathrm{AL} \equiv \lambda_c K/\sqrt{2} \qquad \text{(A4.12)}$$

$$\text{BETA} \;=\; \text{BE} \;\equiv\; \tau_c \eta K^2/\sqrt{2} \qquad \text{(A4.12')}$$

The program carries out the necessary integrations in *lines 530-2080*, with the integrand of (3.165a) being split up into three parts, as described in *section 3.10.2*. The values of the integrals are then combined in *lines 2090-2150* to give

$$\text{RATIO} \;\equiv\; D \text{ , as defined in (3.166)} \qquad \text{(A4.13)}$$

$$\text{RNLDS} \;\equiv\; R'_m \qquad \text{(A4.14)}$$

In *lines 2160-2610*, the program searches for zero crossings in D. If a zero crossing is found, a linear interpolation is performed to locate the zero approximately. The interpolation is carried out between the value of D at (ALFA, BETA) and the value at (ALFA+ALFSTP, BETA), for a given pair of values (ZNOT,YNOT). Finally, the position of the interpolated zero in D is used to give an approximate interpolation between the values of ALFA and RNLDS on either side of the zero crossing *(see lines 2460-2540 and 2320-2340)*.

The output of the program is a tabular display of the form shown overleaf. If a zero crossing has been found in D, the values of D on either side of the zero are displayed as D_1 and D_2. α_{interp} and $(R'_m)_{interp}$ are then the approximate interpolated values of ALFA and RNLDS at the zero in D. If no zero crossing is found,

the values of D and RNLDS at the last pair of ALFA values considered are displayed as $[D_1, D_2]$ and $[(R'_m)_1, (R'_m)_2]$, while $[\alpha_{interp}, (R'_m)_{interp}]$ are set equal to zero.

<u>Form of output</u>

$$\begin{array}{lccc}
 & & \text{BETA} & \\
 & \beta_1 & \beta_2 & \cdots \\
\text{ALFA} & \alpha_1 & \cdots & \\
 & \alpha_{interp} & & \\
 & \alpha_2 & & \\
\text{RNLDS} & (R'_m)_1 & \cdots & \\
 & (R'_m)_{interp} & & \\
 & (R'_m)_2 & & \\
\text{D VALUES} & D_1 & \cdots & \\
 & D_2 & &
\end{array} \qquad (A4.15)$$

The program then asks if iteration is required *(line 2870)*. If an integer N is read in at this point, a successive iteration of the interpolation will be carried out. The values of D and RNLDS corresponding to ALFA = α_{interp} are calculated, and the new value of D (say D_{new}) is used to determine whether the zero in D lies between D_1 and D_{new} or between D_{new} and D_2 .

A new interpolation is then carried out between the appropriate pair of values. The interpolation is iterated N times, and a new set of output values is generated.

If further interpolation is not required, a blank read in at *line 2880* will send the program back to ask for a new set of ALFA and BETA values.

It should be noted that the values of ALFA and BETA presented in the output have been re-converted to the input form, so that

$$\begin{aligned} \text{ALFA} &= \lambda_c/L \\ \text{BETA} &= \tau_c \eta K^2/2\pi \end{aligned} \qquad \text{(A4.16)}$$

The factor 2π is included in the definition of BETA so that

$$\begin{aligned} \text{BETA}\cdot\text{ZNOT} &= [\tau_c \eta K^2/2\pi]\cdot[\text{Re}\ \ \Omega/\eta K^2] \\ &= \tau_c \cdot \text{Re}\ \Omega/2\pi \\ &= \tau_c/T \end{aligned} \qquad \text{(A4.17)}$$

A.4.3 The program

The program described in the last section and in *section 3.10.2* is listed on the next few pages.

```
      REAL*8 GZR(40),GYR(40),WTGZR(40),WTGYR(40)
      DIMENSION RMI(20,3)
C     EXTERNAL SIMDBL
      LOGICAL LV2
      LOGICAL LV1
      INTEGER PL
      DIMENSION AP(20),BETAP(20)
      REAL*8 F(40,40),G(40,40),GZI(40),GYI(40),WTGZI(40),WTGYI(40)
      REAL*8 RATIO(10),RNLDS(10),ALFA(12),BETA(20),YL(15)
       DOUBLE PRECISION ARINUM,ARIDEN,B2A,SCALE,STRET
      DOUBLE PRECISION YNOT,ALFASQ,BETASQ,ZNOT,XLSQ,XHSQ
      DOUBLE PRECISION    YM2Y,YP2Y,YMYP,ARITAN,ARLOG,FACTR,FACTI
      REAL*4 AI(20,3),RI(20,3),D(20,2),T(20)
      XYZ=4.442882938
C     SECTION TO READ INTEGRATION DATA
    1 WRITE(6,936)
  936 FORMAT('TO CONTINUE ENTER BLANK,OTHERWISE EOF')
      READ(5,700,END=40) I
      WRITE(6,703)
  703 FORMAT('TO READ POINTS FOR REAL INTEGRATION ENTER BLANK;',
     1 ' IMAGINARY ONLY ENTER EOF')
      READ(5,700,END=7) I
  700 FORMAT(I2)
      CALL RDATA(GYR,GZR,WTGYR,WTGZR,NR,MR)
    7 WRITE(6,705)
  705 FORMAT('TO READ POINTS FOR IMAG INTEGRATION ENTER BLANK',
     2 ' OTHERWISE EOF')
      READ(5,700,END=5) I
      CALL RDATA(GYI,GZI,WTGYI,WTGZI,NI,MI)
      WRITE(6,400)   NR,MR
      WRITE(6,807)  NI,MI
  400 FORMAT(//' REAL ',I3,'  GAUSSIAN Y POINTS',I3,'  GAUSSIAN',
     1' Z POINTS'//)
  807 FORMAT(//' IMAG ',I3,'  GAUSSIAN Y POINTS',I3,'  GAUSSIAN',
     1' Z POINTS'//)
    3 WRITE(6,420)
```

```
  420 FORMAT(//'NEED TOLERANCES EPY1,EPY4,EPG1,EPG2 AND NO OF STEPS')
      READ(5,421,END=1) EPY1,EPY4,EPG1,EPG2,K
  421 FORMAT(4F10.5,I3)
    5 WRITE(6,399)
  399 FORMAT(///'NEED RE,IM OMEGA STAR/KSQ')
      READ(5,500,END=3)    ZNOT,YNOT
      SLOPE=(YNOT-1.0)/ZNOT
  500 FORMAT(2F20.15)
  401 FORMAT(//'NEED STALF,INCR,#,STBET,INCR,#')
    2 WRITE(6,401)
      READ(5,502,END=5) STALF,ALFSTP,IALF,STBET,BETSTP,IBET
C     SECTION TO MULTIPLY BY PI*ROOT 2
      STALF=STALF*XYZ
      ALFSTP=ALFSTP*XYZ
      STBET=STBET*XYZ
      BETSTP=BETSTP*XYZ
      PL=0
      ML=0
      MP=0
      IT=0
      WRITE(6,743)
  743 FORMAT('INTEGRATION LIMITS? TYPE 1 IF YES')
      READ(5,744) PL
  744 FORMAT(I1)
      WRITE(6,843)
  843 FORMAT('VOLUMES?  TYPE 1 IF YES')
      READ(5,744,END=2)  ML
      WRITE(6,901)
  502 FORMAT(2(2F20.7,I2))
   22 DO 36 I=1,IBET
   36 T(I)=1.
      IF(MP.GE.1) IALF=1
      NZERO=K
      ALFA(1)=STALF
      BETA(1)=STBET
      Y2=DSQRT(YNOT-1.)
```

```
      DO 38 N=1,IBET
      BETA(N+1)=BETA(N)+BETSTP
      BETASQ=BETA(N)*BETA(N)
      DO 37 L=1,IALF
      IF(MP.GE.1) ALFA(L)=AI(N,2)
      IF(ALFA(L).EQ.0.) GO TO 37
      ALFA(L+1)=ALFA(L)+ALFSTP
      ALFASQ=ALFA(L)*ALFA(L)
      AL=ALFA(L)
      BE=BETA(N)
      Y3=2.+DSQRT(YNOT)
      LV1=.TRUE.
      LV2=.FALSE.
      VOL1R=0
C     SECTION TO FIX LIMITS ON REGION BEYOND FIRST + PEAK IN 3.165(A)
      IF(AL*Y3.GT.2.*ZNOT*BE)  LV2=.TRUE.
      IF(((2.*BE*ZNOT)**2+(AL*Y3)**2).GT.100) LV1=.FALSE.
      IF(.NOT.LV1.AND.LV2) GO TO 4
      IF(BE*ZNOT.LT.2.0) GO TO 53
      YINC=.2/AL
      YS=Y3
      CALL EXTFND(AL,BE.ZNOT.YNOT,EXT,ZNOT,0,YS,YINC,9,K)
      IF(EXT.EQ.0) YS=Y3
      TOL=EPY4*EXT
      YINC=2.*YINC
      Y4=YS
      CALL CORY(AL,BE,ZNOT,YNOT,K,TOL,YINC,Y4,ZNOT,9,K)
      G1=ZNOT-2./BE
      G2=ZNOT+2./BE
      GO TO 58
C     SECTION TO INTEGRATE REGION BEYOND FIRST + PEAK IN 3.165(A)
   53 CALL R3(ALFA(L),BETA(N),ZNOT,YNOT,G1,G2,Y3,Y4,EPG1,EPG2,EPY4,K)
   58 CALL SETUPR(ALFA(L),BETA(N),ZNOT,YNOT,GYR,GZR,NR,MR,Y3,Y4,G1,G2,F)
      VOL3R=DBLINT(WTGYR,WTGZR,F,NR,MR)*(Y4-Y3)*(G2-G1)/4
C     IF(.NOT.LV1.AND..NOT.LV2) GO TO 1001
```

```
C       SECTION TO FIX LIMITS ON NEGATIVE-VALUE REGION IN 3.165(A)
      4 YN=DSQRT(YNOT)-1.
        IF(.NOT.LV1.AND.AL*Y3.GT.7.) Y3=7./AL
        YINC=YN/K
        YQ=DSQRT(YNOT-1.)
        YSTP=(Y3-YQ)/K
        GS=0
        CALL EXTFND(AL,BE,ZNOT,YNOT,CFRIN,GS,0,YQ,YSTP,0,K)
        Y1=0
         IF((ZNOT*BETA(N)).GT.0.707) GO TO 65
C       CFRIN=-CFR(ALFA(L),BETA(N),ZNOT,YNOT,0,YN)
        TOL=EPY1*CFRIN
C       CALL CORY(AL,BE,ZNOT,YNOT,NZERO,TOL,YINC,Y1,GS,16,K)
        Y1=0
        IF(LV1) GO TO 8
        G2=GS
        GINC=.2/BE
        CALL CORG(AL,BE,ZNOT,YNOT,NZERO,TOL,GINC,G2,YQ,15,K)
      8 GO TO 64
     65 IF(BE*ZNOT.LT.2.) GO TO 57
        G1=ZNOT-2./BE
        G2=ZNOT+2./BE
        TOL=EPY1*CFR(AL,BE,ZNOT,YNOT,ZNOT,YN)
C       CALL CORY(AL,BE,ZNOT,YNOT,K,TOL,YINC,Y1,ZNOT,18,K)
        Y1=0
        GO TO 56
     57 GESC=ZNOT*1./K
        CALL EXTFND(AL,BE,ZNOT,YNOT,BIG,GS,GESC,YQ,0,18,K)
        Y1=0
        TOL=EPY1*BIG
C       CALL CORY(AL,BE,ZNOT,YNOT,NZERO,TOL,YINC,Y1,GS,18,K)
        Y1=0
        IF(LV1) GO TO 64
        G2=GS
        GINC=.2/BE
        CALL CORG(AL,BE,ZNOT,YNOT,NZERO,TOL,GINC,G2,YQ,13,K)
```

```
      64 IF(.NOT.LV1.AND.LV2) G1=0
C        INTEGRATION OF NEGATIVE-VALUE REGION IN 3.165(A)
      56 CALL SETUPR(ALFA(L),BETA(N),ZNOT,YNOT,GYR,GZR,NR,MR,Y1,Y2,G1,G2,F)
         VOL1R=DBLINT(WTGYR,WTGZR,F,NR,MR)*(Y2-Y1)*(G2-G1)/4
         IF(.NOT.LV1.AND.AL*Y3.GT.7.) Y3=7./AL
C        INTEGRATION OF REGION AROUND FIRST + PEAK IN 3.165(A)
    1001 CALL SETUPR(ALFA(L),BETA(N),ZNOT,YNOT,GYR,GZR,NR,MR,Y2,Y3,G1,G2,F)
         VOL2R=DBLINT(WTGYR,WTGZR,F,NR,MR)*(Y3-Y2)*(G2-G1)/4
         IF(.NOT.LV1.AND.LV2) VOL3R=0
         IF(PL.EQ.1) WRITE(6,365) Y1,Y4,G1,G2
     365 FORMAT(/'LIMITS FOR REAL INTEGRATION',/'    Y1=',E20.8,
        1 '    Y4=',E20.8,'    G1=',E20.8,'    G2=',E20.8)
C        SECTION TO FIND REGION BOUNDS FOR 3.165(B)
         IF(BE*ZNOT.LT.2) GO TO 153
         GS=ZNOT
         IF(1.414.GT.AL*(DSQRT(YNOT)+1.).AND.GS.GT.0.3) GO TO 158
         YESC=.2
         KQ=(DSQRT(YNOT)+1.)/.2+10
         IF(KQ.LT.K) KQ=K
         GO TO 157
     158 YESC=.2/AL
         KQ=K
     157 YS=.0001
         CALL EXTFND(AL,BE,ZNOT,YNOT,EXT,GS,0,YS,YESC,21,KQ)
         Y4=YS
         Y1=0
         TOL=EPY4*EXT
         CALL CORY(AL,BE,ZNOT,YNOT,K,TOL,YESC,Y4,GS,21,K)
         YESC=YS/K
         TOL=EPY1*EXT
         CALL CORY(AL,BE,ZNOT,YNOT,K,TOL,YESC,Y1,GS,23,K)
         GO TO 156
     153 GS=ZNOT+1/BETA(N)
         GESC=GS*3./(2.*K)
         YP=.0001
```

```
      IF(1.414.GT.AL*(DSQRT(YNOT)+1.).AND.GS.GT.0.3) GO TO 27
       YESC=.2
      KQ=(DSQRT(YNOT)+1.)/.2+10
      IF(KQ.LT.K)   KQ=K
      GO TO 28
   27 YESC=.2/AL
      KQ=K
   28 TX=GESC/10.
      TEMP=SFMIN(AL,BE,ZNOT,YNOT,TX,TX,GESC,YESC,GS,YS,YP,KQ)
   31 Y4=YS
      Y1=0
      TOL=EPY4*TEMP
      CALL CORY(AL,BE,ZNOT,YNOT,NZERO,TOL,YESC,Y4,GS,22,K)
      YESC=YS/K
      TOL=EPY1*TEMP
      CALL CORY(AL,BE,ZNOT,YNOT,NZERO,TOL,YESC,Y1,GS,20,K)
      GESC=.2/BE
      TOL=EPG2*TEMP
      G2=GS
      CALL CORG(AL,BE,ZNOT,YNOT,NZERO,TOL,GESC,G2,YS,26,K)
      GESC=GS/K
      G1=0
      TOL=EPG1*TEMP
      CALL CORG(AL,BE,ZNOT,YNOT,NZERO,TOL,GESC,G1,YS,24,K)
C     SECTION TO INTEGRATE 3.165(B)
  156 CALL SETUPI(ALFA(L),BETA(N),ZNOT,YNOT,GYI,GZI,NI,MI,Y1,Y4,G1,G2,G)
      IF(PL.EQ.1) WRITE(6,366) Y1,Y4,G1,G2
  366 FORMAT(/'LIMITS FOR IMAGINARY INTEGRATION',/'   Y1=',E20.8,
     2 '   Y4=',E20.8,'   G1=',E20.8,'   G2=',E20.8)
      VOLI=DBLINT(WTGYI,WTGZI,G,NI,MI)*(Y4-Y1)*(G2-G1)/4
```

```
C       SECTION TO EVALUATE  D  AND  R'm
        RATIO(L)=(VOL1R+VOL2R+VOL3R)/(VOLI*SLOPE)+1
        RN=-ZNOT/((ALFA(L)**5)*VOLI*2.)
        RNLDS(L)=ALFA(L)*DSQRT(RN*6.*3.14159265/BETA(N))
        IF(ML.EQ.1.AND.IT.EQ.MP) WRITE(6,800) VOL1R,VOL2R,VOL3R,VOLI
  800   FORMAT(/'VOL1R=',E20.8,'   VOL2R=',E20.8,'   VOL3R=',E20.8,
     1  '   VOLI=',E20.8/)
   37   CONTINUE

C       SECTION TO CARRY OUT INTERPOLATION
        IF(MP.GE.1) GO TO 159
        M=1
        TEMP=RATIO(1)
   66   M=M+1
        IF(M.GT.IALF) GO TO 59
        IF(RATIO(M)*TEMP)    68,68,67
   67   TEMP=RATIO(M)
        GO TO 66
   68   AI(N,1)=ALFA(M-1)
        AI(N,3)=ALFA(M)
        RI(N,1)=RNLDS(M-1)
        RI(N,3)=RNLDS(M)
        D(N,1)=TEMP
        D(N,2)=RATIO(M)
        TER=TEMP/RATIO(M)
        FR=1./(1.-TER)
   26   AI(N,2)=AI(N,3)-(AI(N,3)-AI(N,1))*FR
        RI(N,2)=RI(N,3)-(RI(N,3)-RI(N,1))*FR
        GO TO 38
   59   NL=1
        AI(N,2)=0.
        AI(N,1)=ALFA(IALF-1)
        AI(N,3)=ALFA(IALF)
        D(N,1)=RATIO(IALF-1)
        D(N,2)=RATIO(IALF)
```

```
      RI(N,2)=0.
      RI(N,1)=RNLDS(IALF-1)
      RI(N,3)=RNLDS(IALF)
      T(N)=0.
      GO TO 38
  159 DD=RATIO(1)
      IF(DD.EQ.0) GO TO 38
      IF(DD*D(N,1))  23,23,25
   23 D(N,2)=DD
      AI(N,3)=AI(N,2)
      RI(N,3)=RI(N,2)
      TER=D(N,1)/DD
      FR=1./(1.-TER)
      GO TO 26
   25 D(N,1)=DD
      AI(N,1)=AI(N,2)
      RI(N,1)=RI(N,2)
      TER=DD/D(N,2)
      FR=1./(1.-TER)
      GO TO 26
   38 CONTINUE
C       OUTPUT SECTION
  901 FORMAT(/40X,'BETA'//)
      DO 29 I=1,IBET
   29 BETAP(I)=BETA(I)/XYZ
      IF(IT.EQ.MP) WRITE(6,902)  (BETAP(N),  N=1,IBET)
  902 FORMAT(14X,8E12.5,//)
      IF(IT.EQ.MP) WRITE(6,908)
  908 FORMAT(//'ALFA')
      DO 343 J=1,3
      DO 32 I=1,IBET
   32 AP(I)=AI(I,J)/XYZ
  343 IF(IT.EQ.MP) WRITE(6,903) (AP(N), N=1,IBET)
      IF(IT.EQ.MP) WRITE(6,909)
  909 FORMAT(//'RNLDS')
```

```
      DO 344 J=1,3
  344 IF(IT.EQ.MP) WRITE(6,904)  (RI(N,J), N=1,IBET)
      IF(IT.EQ.MP) WRITE(6,910)
  910 FORMAT(//'D VALUES')
      DO 345 J=1,2
  345 IF(IT.EQ.MP) WRITE(6,903) (D(N,J),  N=1,IBET)
  904 FORMAT(13X,8E12.5)
  903 FORMAT(13X,8E12.5)
      IF(IT.EQ.MP) GO TO 77
      IT=IT+1
      GO TO 22
   77 WRITE(6,906)
  906 FORMAT(//'ITERATE? HOW MANY TIMES?')
      READ(5,907) MP
      IT=0
  907 FORMAT(I1)
      IF(MP.GE.1) GO TO 22
      GO TO 2
   40 STOP
      END


C     INTEGRATION SUBROUTINE
      FUNCTION DBLINT(WTL,WTH,F,NX,MX)
      DOUBLE PRECISION  WTL(40),WTH(40),F(40,40)
      DOUBLE PRECISION TEMP
      DBLINT=0
      DO 12 I=1,NX
      TEMP=0
      DO  13 J=1,MX
   13 TEMP=TEMP+WTH(J)*F(I,J)
   12 DBLINT=DBLINT+WTL(I)*TEMP
      RETURN
      END
```

```
C       SUBROUTINE TO CALCULATE INTEGRAND IN 3.165(A)
        FUNCTION CFR(ALFA,BETA,ZNOT,YNOT,G,Y)
        REAL*8 ALFASQ,BETASQ,YSQ,F,YM2Y,YP2Y,YMYP,ARINUM,ARIDEN
        REAL*8 ALFA,BETA,ZNOT,YNOT,G,Y
        REAL*8 UPLOG,DENLOG,ARLOG,FACTR,CFR
        REAL*8 EXARG1,EXARG2
        ALFASQ=ALFA*ALFA
        BETASQ=BETA*BETA
        YSQ=Y*Y
        F=1+YSQ-YNOT
        YM2Y=F-2.*Y
        YP2Y=F+2.*Y
        YMYP=YM2Y*YP2Y
        ARINUM=4.*Y*G
        ARIDEN=YMYP+G*G
        UPLOG=(YM2Y*YM2Y+G*G)
        IF(UPLOG.EQ.0) GO TO 34
        DENLOG=(YP2Y*YP2Y+G*G)
        IF(DENLOG.EQ.0) GO TO 34
        ARLOG=UPLOG/DENLOG
        FACTR=YSQ*F+Y*(YMYP-G*G)*DLOG(ARLOG)/8.-Y*G*F*DATAN2(ARINUM,
       1ARIDEN)/2.
        GO TO 70
     34 FACTR=2.*(1.-DSQRT(YNOT))**3
     70 EXARG1=-(ALFASQ*YSQ+BETASQ*(G-ZNOT)*(G-ZNOT))
        IF(EXARG1.LT.-170) EXARG1=-170
        EXARG2=-(ALFASQ*YSQ+BETASQ*(G+ZNOT)*(G+ZNOT))
        IF(EXARG2.LT.-170) EXARG2=-170
        CFR=FACTR*(DEXP(EXARG1)+DEXP(EXARG2))/2.
        RETURN
        END
C       SUBROUTINE TO CALCULATE INTEGRAND IN 3.165(B)
        FUNCTION SFI(ALFA,BETA,ZNOT,YNOT,G,Y)
        REAL*8 ALFASQ,BETASQ,YSQ,F,YM2Y,YP2Y,YMYP,ARINUM,ARIDEN
        REAL*8 UPLOG,DENLOG,ARLOG,FACTI,SFI
        REAL*8 ALFA,BETA,ZNOT,YNOT,G,Y
```

```
      REAL*8 EXARG1,EXARG2
      ALFASQ=ALFA*ALFA
      BETASQ=BETA*BETA
      YSQ=Y*Y
      F=1+YSQ-YNOT
      YM2Y=F-2.*Y
      YP2Y=F+2.*Y
      YMYP=YM2Y*YP2Y
      ARINUM=4.*Y*G
      ARIDEN=YMYP+G*G
      UPLOG=(YM2Y*YM2Y+G*G)
      IF(UPLOG.EQ.0) GO TO 34
      DENLOG=(YP2Y*YP2Y+G*G)
      IF(DENLOG.EQ.0) GO TO 34
      ARLOG=UPLOG/DENLOG
      FACTI=YSQ*G+Y*G*F*DLOG(ARLOG)/4.+Y*(YMYP-G*G)*DATAN2(ARINUM,
     2ARIDEN)/4.
      GO TO 70
   34 FACTI=0
   70 EXARG1=-(ALFASQ*YSQ+BETASQ*(G-ZNOT)*(G-ZNOT))
      IF(EXARG1.LT.-170) EXARG1=-170
      EXARG2=-(ALFASQ*YSQ+BETASQ*(G+ZNOT)*(G+ZNOT))
      IF(EXARG2.LT.-170) EXARG2=-170
      SFI=FACTI*(DEXP(EXARG1)-DEXP(EXARG2))/2.
      RETURN
      END
C     OVERFLOW SUBROUTINE
      SUBROUTINE ALARM(G,Y,EX,N)
      WRITE(6,34) G,Y,N
   34 FORMAT(//'ALARM AT G=',E20.8,'    Y=',E20.8,'    N=',I3)
      EP=-180
      RETURN
      END
```

```
C     EXTREMUM-FINDING SUBROUTINE
      SUBROUTINE EXTFND(AL,BE,ZNOT,YNOT,EXT,GS,GESC,YS,YESC,NUMB,K)
      EXTERNAL CFR,SFI
      SLOPE=0
      Y=YS
      G=GS
      N=0
      IF(NUMB.GT.19) GO TO 100
      TEMP=ABS(CFR(AL,BE,ZNOT,YNOT,GS,YS))
    7 Y=Y+YESC
      G=G+GESC
      N=N+1
      IF(N.GT.K) GO TO 300
      DIFF=ABS(CFR(AL,BE,ZNOT,YNOT,G,Y))-TEMP
      IF((SLOPE*DIFF).LT.0) GO TO 200
      TEMP=TEMP+DIFF
      IF(DIFF.GT.0) SLOPE=DIFF
      GO TO 7
  100 TEMP=-SFI(AL,BE,ZNOT,YNOT,GS,YS)
  500 FORMAT('IMAGINARY PART =',E20.8)
    8 Y=Y+YESC
      N=N+1
      G=G+GESC
      IF(N.GT.K) GO TO 300
      DIFF=-(SFI(AL,BE,ZNOT,YNOT,G,Y)+TEMP)
      IF((SLOPE*DIFF).LT.0) GO TO 200
      TEMP=TEMP+DIFF
      IF(DIFF.GT.0) SLOPE=DIFF
      GO TO 8
  300 EXT=0
      GO TO 400
  200 EXT=TEMP
      YS=Y-YESC
      GS=G-GESC
  400 RETURN
      END
```

```
C       INTERPOLATION SUBROUTINE FOR USE WITH (B) INTEGRAND
        FUNCTION SSFI(AL,BE,ZN,YN,Y,DY,G,DG)
        EXTERNAL SFI
        YPDY=Y+DY
        YMDY=Y-DY
        GPDG=G+DG
        GMDG=G-DG
        FIRST=SFI(AL,BE,ZN,YN,GPDG,YPDY)
        SECND=SFI(AL,BE,ZN,YN,GMDG,YMDY)
        SSFI=(FIRST-SECND)/(2.*SQRT(DG*DG+DY*DY))
        RETURN
        END
C       SUBROUTINE TO FIX LIMITS ON REGION BEYOND FIRST + PEAK IN 3.165(A)
        SUBROUTINE R3(ALFA,BETA,ZNOT,YNOT,G1,G2,Y3,Y4,EPG1,EPG2,EPY4,K)
        EXTERNAL CFR,SFI
        NZERO=K
        G1=0
        YINC=.2/ALFA
        GINC=.2/BETA
        IF((ZNOT*BETA).GT.0.707) GO TO 70
        IF((2.*ALFA*(SQRT(YNOT)+1)).GT.1.0) GO TO 60
        YM=1/ALFA
        Y4=YM
        CFRBIG=CFR(ALFA,BETA,ZNOT,YNOT,0,YM)
    701 FORMAT('REAL PART=', E20.8)
        GS=0
        TOL=EPY4*CFRBIG
        CALL CORY(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,YINC,Y4,GS,2,K)
        G2=0
        TOL=EPG2*CFRBIG
        CALL CORG(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,GINC,G2,YM,4,K)
        GO TO 101
     60 GS=0
        Y4=Y3
        CFRBIG=CFR(ALFA,BETA,ZNOT,YNOT,0,Y3)
```

```
      TOL=EPY4*CFRBIG
      CALL CORY(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,YINC,Y4,GS,6,K)
      G2=0
      TOL=EPG2*CFRBIG
      CALL CORG(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,GINC,G2,Y3,8,K)
      GO TO 101
   70 GS=ZNOT
      YS=Y3
   80 CALL EXTFND(ALFA,BETA,ZNOT,YNOT,EXT,GS,0,YS,YINC,10,K)
   72 IF(EXT.EQ.0) YS=Y3
      KK=2*K
      GESC=ZNOT*1./K
      GS=0
      CALL EXTFND(ALFA,BETA,ZNOT,YNOT,EXT,GS,GESC,YS,0.12,KK)
      IF(EXT.EQ.0) CALL TRUBL(YS,GS,12,.TRUE.)
      G2=GS
      TOL=EPG2*EXT
      GINC=.2/BETA
      CALL CORG(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,GINC,G2,YS,14,K)
      G1=0
      YINC=.2/ALFA
      TOL=EPY4*EXT
      YINC=2.*YINC
      Y4=YS
      CALL CORY(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,YINC,Y4,GS,10,K)
  101 RETURN
      END
C     WARNING SUBROUTINE
      SUBROUTINE TRUBL(Y,G,NUMB,L)
      LOGICAL L
      IF(L) GO TO 7
      WRITE(6,300) NUMB,Y,G
  300 FORMAT('COORDINATE NOT FOUND IN',I3,/'WHEN K LIMIT REACHED Y=',
     1E20.8,'     G=',E20.8)
      GO TO 8
    7 WRITE(6,400) NUMB,Y,G
```

```
  400 FORMAT('EXTREMUM NOT FOUND IN',I3,/'WHEN K LIMIT REACHED Y=',
     1E20.8,'      G=',E20.8)
    8 RETURN
      END
C     SUBROUTINE TO SET UP INTEGRATION IN 3.165(A)
      SUBROUTINE SETUPR(ALFA,BETA,ZNOT,YNOT,YY,GG,M,NN,A,B,C,D,F)
      EXTERNAL CFR
      REAL*8 F(40,40),YY(40),GG(40)
      DO 13 I=1,M
      DO 13 J=1,NN
      Y=YY(I)*(B-A)/2+(B+A)/2
      H=GG(J)*(D-C)/2+(D+C)/2
   13 F(I,J)=CFR(ALFA,BETA,ZNOT,YNOT,H,Y)
      RETURN
      END
C     SUBROUTINE TO SET UP INTEGRATION IN 3.165(B)
      SUBROUTINE SETUPI(ALFA,BETA,ZNOT,YNOT,YY,GG,M,NN,A,B,C,D,F)
      EXTERNAL SFI
      REAL*8 F(40,40),YY(40),GG(40)
      DO 13 I=1,M
      DO 13 J=1,NN
      Y=YY(I)*(B-A)/2+(B+A)/2
      H=GG(J)*(D-C)/2+(D+C)/2
   13 F(I,J)=SFI(ALFA,BETA,ZNOT,YNOT,H,Y)
      RETURN
      END
C     SUBROUTINE TO FIND PEAK IN INTEGRAND OF 3.165(B)
      FUNCTION SFMIN(AL,BE,ZN,YN,DY,DG,GESC,YESC,GS,YS,YST,K)
      EXTERNAL SFI,SSFI
    1 G=0
      M=0
      SGO=-1.
   13 Y=YST
      G=G+GESC
      SYO=SSFI(AL,BE,ZN,YN,Y,DY,G,0.)
      N=0
```

```
   14 Y=Y+YESC
      SY=SSFI(AL,BE,ZN,YN,Y,DY,G,0.)
      IF(SY*SYO.GT.0) GO TO 26
      Y=Y-SY*YESC/(SY-SYO)
      SG=SSFI(AL,BE,ZN,YN,Y,0.,G,DG)
      M=M+1
      IF(SG*SGO.GT.0) GO TO 27
      G=G-SG*GESC/(SG-SGO)
      GO TO 300
   27 IF(M.GT.K) GO TO 46
      SGO=SG
      GO TO 13
   26 SYO=SY
      N=N+1
      IF(N.GT.K) GO TO 46
      GO TO 14
   46 CALL TRUBL(Y,G,20,.TRUE.)
      READ(5,402,END=301) K
  402 FORMAT(I3)
      GO TO 1
  300 YS=Y
      GS=G
      SFMIN=-SFI(AL,BE,ZN,YN,G,Y)
  301 RETURN
      END
C     SUBROUTINE TO READ INTEGRATION DATA
      SUBROUTINE RDATA(GY,GZ,WTGY,WTGZ,NN,M)
      REAL*8 GY(40),GZ(40),WTGY(40),WTGZ(40)
    1 WRITE(6,200)
  200 FORMAT('NEED EVEN NUMBER OF Y GAUSSIAN POINTS')
      READ(5,201)  NN
  201 FORMAT(I2)
      DO  20  I=1,NN,2
      READ(5,202)     GY(I),WTGY(I)
   20 CONTINUE
      DO 16  J=1,NN,2
```

```
      GY(J+1)=-GY(J)
   16 WTGY(J+1)=WTGY(J)
  202 FORMAT(2F30.15)
      WRITE(6,300)
  300 FORMAT('NEED EVEN NUMBER OF Z GAUSSIAN POINTS')
      READ(5,201)     M
      DO 30 I=1,M,2
      READ(5,202)      GZ(I),WTGZ(I)
   30 CONTINUE
      DO 10  J=1,M,2
      GZ(J+1)=-GZ(J)
   10 WTGZ(J+1)=WTGZ(J)
      RETURN
      END
C     SUBROUTINE TO FIND INTEGRATION LIMIT IN  ξ  DIRECTION
      SUBROUTINE CORY(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,YINC,Y1,G,NUMB,K)
      EXTERNAL CFR,SFI
      N=0
      KC=0
      IF(NUMB.GT.19) GO TO 21
      S1=ABS(CFR(ALFA,BETA,ZNOT,YNOT,G,Y1))-TOL
      GO TO 43
   21 S1=ABS(SFI(ALFA,BETA,ZNOT,YNOT,G,Y1))-TOL
   43 IF(S1.EQ.0) GO TO 100
   27 Y2=Y1+YINC
      IF(NUMB.GT.19) GO TO 61
      S2=ABS(CFR(ALFA,BETA,ZNOT,YNOT,G,Y2))-TOL
      GO TO 44
   61 S2=ABS(SFI(ALFA,BETA,ZNOT,YNOT,G,Y2))-TOL
   44 IF(S1*S2)   33,99,34
   34 S1=S2
      Y1=Y2
      KC=KC+1
      IF(KC.EQ.K) GO TO 18
      GO TO 27
   18 CALL TRUBL(Y1,G,NUMB,.FALSE.)
```

```
      GO TO 100
   33 N=N+1
      EPS=YINC/(1-S1/S2)
      YZERO=Y2-EPS
      IF(NUMB.GT.19)  GO TO 71
      SNOT=ABS(CFR(ALFA,BETA,ZNOT,YNOT,G,YZERO))-TOL
      GO TO 45
   71 SNOT=ABS(SFI(ALFA,BETA,ZNOT,YNOT,G,YZERO))-TOL
   45 IF((NZERO-N).GT.0)  GO TO 47
      Y1=YZERO
      GO TO 100
   47 IF(SNOT*S2) 50,49,51
   49 Y1=YZERO
      GO TO 100
   51 S2=SNOT
      Y2=YZERO
      YINC=YINC-EPS
      GO TO 33
   50 S1=SNOT
      YINC=EPS
      GO TO 33
   99 Y1=Y2
  100 RETURN
      END
C     SUBROUTINE TO FIND INTEGRATION LIMIT IN  ν  DIRECTION
      SUBROUTINE CORG(ALFA,BETA,ZNOT,YNOT,NZERO,TOL,GINC,G1,Y,NUMB,K)
      EXTERNAL CFR,SFI
      N=0
      KC=0
      IF(NUMB.GT.19) GO TO 21
      S1=ABS(CFR(ALFA,BETA,ZNOT,YNOT,G1,Y))-TOL
      GO TO 43
   21 S1=ABS(SFI(ALFA,BETA,ZNOT,YNOT,G1,Y))-TOL
   43 IF(S1.EQ.0) GO TO 100
   27 G2=G1+GINC
      IF(NUMB.GT.19) GO TO 61
```

```
      S2=ABS(CFR(ALFA,BETA,ZNCT,YNOT,G2,Y))-TOL
      GO TO 44
   61 S2=ABS(SFI(ALFA,BETA,ZNOT,YNOT,G2,Y))-TOL
   44 IF(S1*S2)  33,99,34
   34 S1=S2
      G1=G2
      KC=KC+1
      IF(KC.EQ.K) GO TO 18
      GO TO 27
   18 CALL TRUBL(Y,G1,NUMB,.FALSE.)
      GO TO 100
   33 N=N+1
      EPS=GINC/(1-S1/S2)
      GZERO=G2-EPS
      IF(NUMB.GT.19)  GO TO 71
      SNOT=ABS(CFR(ALFA,BETA,ZNOT,YNOT,GZERO,Y))-TOL
      GO TO 45
   71 SNOT=ABS(SFI(ALFA,BETA,ZNOT,YNOT,GZERO,Y))-TOL
   45 IF((NZERO-N).GT.0)  GO TO 47
      G1=GZERO
      GO TO 100
   47 IF(SNOT*S2) 50,49,51
   49 G1=GZERO
      GO TO 100
   51 S2=SNOT
      G2=GZERO
      GINC=GINC-EPS
      GO TO 33
   50 S1=SNOT
      GINC=EPS
      GO TO 33
   99 G1=G2
  100 RETURN
      END
```

A.4.4 Gaussian points and weights

The Gaussian integration points and weights required by the program listed in the last section can be stored in a particularly convenient form, as shown below. If the data is kept in a file "PTSWTS", the instruction

CONTINUE WITH PTSWTS(800,804) RETURN (A4.18)

will provide the program with the information required for an n-point integration with n = 8 . Similarly, if PTSWTS (1600,1608) is used in place of PTSWTS(800,804), the program will be given the information required for an integration with n = 16 . The points and weights listed are taken from *Abramowitz and Stegun (1964)*.

```
400     4.
401     0.339981043584856,0.652145154862546,
402     0.861136311594053,0.347854845137454,
800     8.
801     0.183434642495650,0.362683783378362,
802     0.525532409916329,0.313706645877887,
803     0.796666477413627,0.222381034453374,
804     0.960289856497536,0.101228536290376,
1200    12.
1201    0.125233408511469,0.249147045813403,
1202    0.367831498998180,0.233492536538355,
1203    0.587317954286617,0.203167426723066,
1204    0.769902674194305,0.160078328543346,
1205    0.904117256370475,0.106939325995318,
1206    0.981560634246719,0.047175336386512,
```

1600	16,
1601	0.095012509837637440185,0.189450610455068496285,
1602	0.281603550779258913230,0.182603415044923588867,
1603	0.458016777657227386342,0.169156519395002538189,
1604	0.617876244402643748447,0.149595988816576732081,
1605	0.755404408355003033895,0.124628971255533872052,
1606	0.865631202387831743880,0.095158511682492784810,
1607	0.944575023073232576078,0.062253523938647892863,
1608	0.989400934991649932596,0.027152459411754094852,
2000	20,
2001	0.076526521133497333755,0.152753387130725850698,
2002	0.227785851141645078080,0.149172986472603746788,
2003	0.373706088715419560673,0.142096109318382051329,
2004	0.510867001950827098004,0.131688638449176626898,
2005	0.636053680726516025453,0.118194531961518417312,
2006	0.746331906460150792614,0.101930119817240435037,
2007	0.839116971822218823395,0.083276741576704748725,
2008	0.912234428251325905868,0.062672048334109063570,
2009	0.963971927277913791268,0.040601429800386941331,
2010	0.993128599185094924786,0.017614007139152118312,
2400	24,
2401	0.064056892862605626085,0.127938195346752156974,
2402	0.191118867473616309159,0.125837456346828296121,
2403	0.315042679696163374387,0.121670472927803391204,
2404	0.433793507626045138487,0.115505668053725601353,
2405	0.545421471388839535658,0.107444270115965634783,
2406	0.648093651936975569252,0.097618652104113888270,
2407	0.740124191578554364244,0.086190161531953275917,
2408	0.820001985973902921954,0.073346481411080305734,
2409	0.886415527004401034213,0.059298584915436780746,
2410	0.938274552002732758524,0.044277438817419806169,
2411	0.974728555971309498198,0.028531388628933663181,
2412	0.995187219997021360180,0.012341229799987199547,
3200	32,
3201	0.048307665687738316235,0.096540088514727800567,
3202	0.144471961582796493485,0.095638720079274859419,
3203	0.239287362252137074545,0.093844399080804565639,
3204	0.331868602282127649780,0.091173878695763884713,
3205	0.421351276130635345364,0.087652093004403811143,
3206	0.506899908932229390024,0.083311924226946755222,
3207	0.587715757240762329041,0.078193895787070306472,
3208	0.663044266930215200975,0.072345794108848506225,
3209	0.732182118740289680387,0.065822222776361846838,
3210	0.794483795967942406963,0.058684093478535547145,
3211	0.849367613732569970134,0.050998059262376176196,
3212	0.896321155766052123965,0.042835898022226680657,
3213	0.934906075937739689171,0.034273862913021433103,
3214	0.964762255587506430774,0.025392065309262059456,
3215	0.985611511545268335400,0.016274394730905670605,
3216	0.997263861849481563545,0.007018610009470096600,